

A priori error estimates of the DtN-FEM: fluid-solid interaction problems

Tao Yin (taoyin_cqu@163.com)

College of Mathematics and Statistics, Chongqing University, Chongqing 401331, China

Liwei Xu (xul@cqu.edu.cn)

College of Mathematics and Statistics, Chongqing University, Chongqing 401331, China

Institute of Computing and Data Sciences, Chongqing University, Chongqing 400044, China

Abstract

We consider the finite element method solving a fluid-solid interaction (FSI) problem in two dimensions. The original problem is reduced to an equivalent nonlocal boundary value problem through an exact Dirichlet-to-Neumann (DtN) mapping defined on an artificial boundary enclosing the solid. The solvability results are established for the corresponding variational problem and its modified form resulting from truncation of the DtN mapping. Regarding to the numerical solutions, we derive a priori error estimates involving the effects of both finite element discretization and infinite series truncation. Numerical examples are presented to illustrate the accuracy of numerical schemes and validate the theoretical results.

Keywords: Dirichlet-to-Neumann mapping, finite element method, fluid-solid interaction problem, error analysis.

1 Introduction

In this work, we consider numerical solutions of the time-harmonic fluid-solid interaction (FSI) problem, the scattering of a time-harmonic acoustic incident wave by an elastic body immersed in a fluid. This problem plays an important role in many fields of application, including exploration seismology, oceanography, and non-destructive testing, and to name a few. We consider the linear, isotropic and bounded elastic body whose displacement field is modeled by the time-harmonic Navier equation. The medium outside the elastic body is filled with a homogeneous compressible inviscid fluid and the acoustic scattered field is described by the Helmholtz equation together with the Sommerfeld's radiation condition at infinity.

It has been studied extensively in the past several decades to develop efficient and accurate numerical methods for the solution of time-harmonic FSI problems. One of the popular methods is the boundary integral equation (BIE) method ([32, 38]) which leads us to solving a system of boundary integral equations defined on the transmission boundary. In addition to the well-known advantages associated with BIE methods, there also exist some restrictions, such as eigenvalue issues and hyper-singularity features associated with boundary integral operators, which need to be extraordinarily considered for this method. The second method is the coupling methods, for instance, using the boundary element method (BEM) to solve the acoustic problem while the finite element methods (FEM) is employed to approximate the interior elastic wave leads to a coupled FEM-BEM method ([3, 6, 7, 25, 33, 40]). Advantages of this method include the feasibility of dealing with elastic mediums with variable coefficients, comparatively

less requirement on the shape of artificial boundary and the fact of naturally and exactly taking into account the Sommerfeld's radiation condition, and to name a few. The third way is to introduce an artificial boundary or an absorbing layer so that one can reduce the original unbounded problem to a nonlocal boundary value problem which could be solved using field equation solvers such as the FEM. In this case, proper absorbing or non-reflected boundary conditions are required to be defined on the artificial boundaries or layers. There are mainly two strategies for the definition of such artificial boundary conditions applied to the exterior time-harmonic acoustic scattering. One is called the perfectly matched layer ([2, 4, 5]), and the other is the Dirichlet-to-Neumann (DtN) mapping based on Fourier series expansions ([9, 10, 31]). Coupling of the DtN mapping and the FEM leads to the so-called DtN-FEM ([17]) which is the method we use in this work to solve the FSI problem.

There exist some issues for the DtN-FEM solving the scattering problem. The first one is the limitation on the shape of the artificial boundary, and it has been considered in earlier works ([35, 36]) where the authors apply perturbation methods to the DtN mapping so that it can be defined on a perturbed curve of the artificial boundary with regular shape such as a circle or an ellipse. The second issue is the potential loss of uniqueness for the solution of modified nonlocal boundary value problem due to the truncation of infinite series of DtN mapping. In [18], the authors defined a modified form of the DtN mapping to circumvent this difficulty. This modified form of DtN mapping is equivalent to the original DtN mapping, and however is not equivalent to the one after truncation ([21]). From the numerical point of view, authors of [22] suggested to make a choice of $N \geq kR$, where N is the truncation order of infinite series, k is the wave number and R is the radius of circular artificial boundary. The third problem is related to the error estimate due to truncation of the DtN mapping, and the errors are indeed understood ([36]) to have an order of exponential decay. For problems of Laplace and linear elasto-statics, authors of [19, 20] have derived a priori error estimates indicating such effects based on the fact that the corresponding sesquilinear form is positive definite. However, it is nontrivial to apply the technique in [19, 20] to the acoustic and elastic scattering problems since the corresponding sesquilinear form is usually indefinite. In this work, we take the FSI problem as the model problem to investigate the last two issues mentioned above for the DtN-FEM. Without doing modification, we will give a novel and strait-forward procedure to prove the unique solvability (Section 4.2) of the truncated variational equation for FSI problem instead of a proof of contradiction in ([27]) where the exterior acoustic scattering problem was considered. In order to investigate the errors due to the truncation of the infinite series DtN mapping, we derive a new and more apparent truncation error estimates declaring exponential attenuation between the exact DtN mapping and its truncated one (Theorem 4.2). Based on these two results, we are allowed to derive a priori error estimates (Theorem 5.3 and 5.4) involving the effects of both finite element discretization and infinite series truncation, and in particular, explicitly indicating the order of exponential decay due to the truncation of DtN mapping.

The remainder of the paper is organized as follows. We first describe the classical FSI problem in Section 2, and then reduce the transmission problem to an equivalent nonlocal boundary value problem in Section 3. Essential mathematical analysis for the corresponding variational equation of the nonlocal boundary value problem and its modification due to truncation of the DtN mapping are discussed in Section 3 and 4, respectively. In Section 5, we establish a priori error estimates including both effects of numerical discretization and for the finite element solution of the modified variational equation. Several numerical experiments are presented to confirm our theoretical results in Section 6 and finally we make a conclusion in Section 7.

2 Statement of the problem

Let $\Omega \subset \mathbb{R}^2$ denote a bounded and simply connected domain with sufficiently smooth interface $\Gamma = \partial\Omega$, and $\Omega^c = \mathbb{R}^2 \setminus \bar{\Omega} \subset \mathbb{R}^2$ be the unbounded exterior region. The domain Ω is occupied by a linear and isotropic elastic solid determined through the Lamé constants λ and μ ($\mu > 0$, $\lambda + \mu > 0$) and its mass density $\rho > 0$, and Ω^c is filled with compressible, inviscid fluids. We denote by $c_0 > 0$ the speed of sound in the fluid, $\rho_f > 0$ the density of the fluid, ω the frequency and $k = \omega/c_0$ the wave number. The

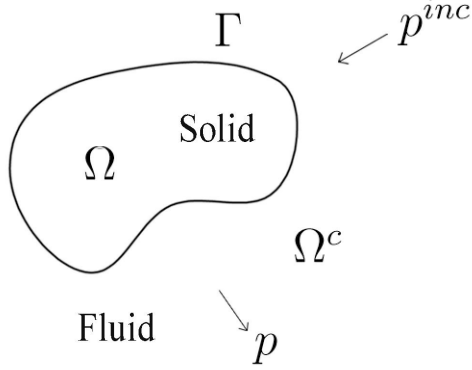


Figure 1: The geometry settings for the fluid-solid interaction problem (2.1)-(2.5).

fluid-solid interaction problem to be considered in this work is to determine the elastic displacement \mathbf{u} in the solid and the acoustic scattered field p in the fluid provided an incident field p^{inc} . It can be described mathematically as follows. Given p^{inc} , find $\mathbf{u} = (u_x, u_y) \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^2$ and $p \in C^2(\Omega^c) \cap C^1(\overline{\Omega^c})$ satisfying

$$\Delta^* \mathbf{u} + \rho \omega^2 \mathbf{u} = \mathbf{0} \quad \text{in } \Omega, \quad (2.1)$$

$$\Delta p + k^2 p = 0 \quad \text{in } \Omega^c, \quad (2.2)$$

$$\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} = \frac{\partial}{\partial \mathbf{n}}(p + p^{inc}) \quad \text{on } \Gamma, \quad (2.3)$$

$$\mathbf{T} \mathbf{u} = -\mathbf{n}(p + p^{inc}) \quad \text{on } \Gamma, \quad (2.4)$$

and the Sommerfeld radiation condition

$$\lim_{r \rightarrow \infty} r^{\frac{1}{2}} \left(\frac{\partial p}{\partial r} - i k p \right) = 0, \quad r = |x|, \quad (2.5)$$

uniformly with respect to all $\hat{x} = x/|x| \in \mathbb{S} := \{x \in \mathbb{R}^2 : |x| = 1\}$. Here, $\partial/\partial \mathbf{n}$ is the normal derivative on Γ (here and in the sequel, \mathbf{n} is always the outward unit normal to the boundary), $i = \sqrt{-1}$ is the imaginary unit. Δ^* is the operator defined by

$$\Delta^* = \mu \Delta + (\lambda + \mu) \nabla \nabla \cdot,$$

and the standard stress operator \mathbf{T} is defined by

$$\mathbf{T} \mathbf{u} = 2\mu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} + \lambda \mathbf{n} \nabla \cdot \mathbf{u} + \mu \mathbf{n} \times \text{curl } \mathbf{u}.$$

It is known ([30]) that the problem (2.1)-(2.5) is not always uniquely solvable due to the occurrence of traction free oscillations for certain geometries and some frequencies ω . These ω are also known as the Jones frequencies which are inherent to the original model. We conclude from [32] the following uniqueness result.

Theorem 2.1. *If the surface Γ and the material parameters (μ, λ, ρ) are such that there are no traction free solutions, the boundary value problem (2.1)-(2.5) has at most one solution. Here, we call a nontrivial \mathbf{u}_0 a traction free solution if it solves*

$$\Delta^* \mathbf{u}_0 + \rho \omega^2 \mathbf{u}_0 = \mathbf{0} \quad \text{in } \Omega,$$

$$\mathbf{T} \mathbf{u}_0 = \mathbf{0} \quad \text{on } \Gamma,$$

$$\mathbf{u}_0 \cdot \mathbf{n} = 0 \quad \text{on } \Gamma.$$

To simplify the presentation throughout the dissertation, we shall denote by $c > 0$, $\alpha > 0$ and $\beta \geq 0$ generic constants whose precise values are not required and may change line by line.

3 Nonlocal boundary value problem

3.1 DtN mapping

We introduce an artificial circular boundary $\Gamma_R := \{x \in \mathbb{R}^2 : |x| = R\}$ such that $\overline{\Omega} \subset B_R := \{x \in \mathbb{R}^2 : |x| < R\}$. The artificial boundary decomposes the exterior domain Ω^c into two subdomains. One is the annular region $\Omega_R = B_R \setminus \overline{\Omega}$ between Γ and Γ_R , and the other is the unbounded exterior region $\Omega_R^c = \mathbb{R}^2 \setminus \overline{B_R}$. On Γ_R , we impose the exact transparent boundary condition

$$\frac{\partial p}{\partial \mathbf{n}} = Sp \quad \text{on } \Gamma_R, \quad (3.1)$$

where S is known as the DtN mapping from $H^s(\Gamma_R)$ to $H^{s-1}(\Gamma_R)$ defined by

$$S\varphi := \sum_{n=0}^{\infty} \frac{kH_n^{(1)'}(kR)}{\pi H_n^{(1)}(kR)} \int_0^{2\pi} \varphi(R, \phi) \cos(n(\theta - \phi)) d\phi \quad \text{for all } \varphi \in H^s(\Gamma_R), 1/2 \leq s \in \mathbb{R}, \quad (3.2)$$

or equivalently,

$$S\varphi := \sum_{n \in \mathbb{Z}} \frac{kH_n^{(1)'}(kR)}{2\pi H_n^{(1)}(kR)} \int_0^{2\pi} \varphi(R, \phi) e^{in(\theta - \phi)} d\phi \quad \text{for all } \varphi \in H^s(\Gamma_R), 1/2 \leq s \in \mathbb{R}. \quad (3.3)$$

Here and in the sequel, the prime behind the summation means that the first term in the summation is multiplied by $1/2$. The transparent boundary condition (3.1) defines a nonlocal condition for p on Γ_R since the Dirichlet data p over the entire boundary Γ_R are required to compute the Neumann data $\partial p / \partial \mathbf{n}$ at a single point $x \in \Gamma_R$. We have the following mapping property for S (see Theorem 3.1 in [27]).

Theorem 3.1. *For any constant $s \geq 1/2$, the DtN mapping S defined by (3.2) or (3.3) is a bounded linear operator from $H^s(\Gamma_R)$ to $H^{s-1}(\Gamma_R)$, that is, there exists a constant $c > 0$ independent of $\varphi \in H^s(\Gamma_R)$ such that*

$$\|S\varphi\|_{H^{s-1}(\Gamma_R)} \leq c \|\varphi\|_{H^s(\Gamma_R)}. \quad (3.4)$$

3.2 Reduced problem

Now, the transmission problem (2.1)-(2.5) can be equivalently replaced by the following nonlocal boundary value problem: Given p^{inc} , find $\mathbf{u} \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^2$ and $p \in C^2(\Omega_R) \cap C^1(\overline{\Omega_R})$ such that

$$\Delta^* \mathbf{u} + \rho \omega^2 \mathbf{u} = \mathbf{0} \quad \text{in } \Omega, \quad (3.5)$$

$$\Delta p + k^2 p = 0 \quad \text{in } \Omega_R, \quad (3.6)$$

$$\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} = \frac{\partial}{\partial \mathbf{n}} (p + p^{inc}) \quad \text{on } \Gamma, \quad (3.7)$$

$$\mathbf{T} \mathbf{u} = -\mathbf{n}(p + p^{inc}) \quad \text{on } \Gamma, \quad (3.8)$$

$$\frac{\partial p}{\partial \mathbf{n}} = Sp \quad \text{on } \Gamma_R. \quad (3.9)$$

The uniqueness result for the nonlocal boundary value problem (3.5)-(3.9) is established in the next Theorem.

Theorem 3.2. *If the surface Γ and the material parameter (μ, λ, ρ) are such that there are no traction free solutions, the nonlocal boundary value problem (3.5)-(3.9) has at most one solution.*

Proof. It is sufficient to prove that the corresponding homogeneous boundary value problem of (3.5)-(3.9) has only the trivial solution. Suppose that (\mathbf{u}_0, p_0) is the solution of the corresponding homogeneous value problem of (3.5)-(3.9). Now let p_1 be the solution of the following exterior Dirichlet problem for the Helmholtz equation

$$\begin{aligned} \Delta p_1 + k^2 p_1 &= 0 & \text{in } \Omega_R^c, \\ p_1 &= p_0 & \text{on } \Gamma_R, \\ \lim_{r \rightarrow \infty} r^{\frac{1}{2}} \left(\frac{\partial p_1}{\partial r} - ikp_1 \right) &= 0, & r = |x|. \end{aligned} \quad (3.10)$$

Then p_1 has the explicit representation form

$$p_1(r, \theta) = \sum_{n \in \mathbb{Z}} a_n H_n^{(1)}(kr) e^{in\theta}, \quad \forall r \geq R, \quad \theta \in [0, 2\pi). \quad (3.11)$$

Calculating the normal derivative of (3.11) and taking the limit as $r \rightarrow R$, we obtain on Γ_R ,

$$\begin{aligned} \frac{\partial p_1}{\partial \mathbf{n}} &= \sum_{n \in \mathbb{Z}} \frac{k H_n^{(1)'}(kR)}{2\pi H_n^{(1)}(kR)} \int_0^{2\pi} p_1(R, \theta) e^{in(\theta-\phi)} d\phi \\ &= S p_1 \\ &= S p_0 \end{aligned}$$

because of the boundary condition (3.10). In addition, the nonlocal boundary condition (3.9) gives

$$\frac{\partial p_0}{\partial \mathbf{n}} = S p_0 \quad \text{on } \Gamma_R.$$

Therefore,

$$\frac{\partial p_1}{\partial \mathbf{n}} - \frac{\partial p_0}{\partial \mathbf{n}} = S p_0 - S p_0 = 0 \quad \text{on } \Gamma_R. \quad (3.12)$$

If we define the function $(\mathbf{u}, p) \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^2 \times (C^2(\Omega_R \cup \Omega_R^c) \cap C^1(\overline{\Omega_R}))$ as

$$(\mathbf{u}, p) = \begin{cases} (\mathbf{u}_0, p_0) & \text{in } \Omega \times \Omega_R, \\ (\mathbf{u}_0, p_1) & \text{in } \Omega \times \Omega_R^c, \end{cases}$$

then by (3.10) and (3.12), we can see that both p and $\partial p / \partial \mathbf{n}$ are continuous across the interface Γ_R . Hence, (\mathbf{u}, p) is the solution of the corresponding homogeneous boundary value problem of (2.1)-(2.5). Then the assumption leads to

$$(\mathbf{u}_0, p_0) = (\mathbf{0}, 0).$$

This completes the proof. \square

3.3 Weak formulation

Now we study the weak formulation of (3.5)-(3.9) which reads: Given p^{inc} , find $\mathbf{U} = (\mathbf{u}, p) \in \mathcal{H}^1 = (H^1(\Omega))^2 \times H^1(\Omega_R)$ such that

$$A(\mathbf{U}, \mathbf{V}) = B(\mathbf{U}, \mathbf{V}) + b(p, q) = \ell(\mathbf{V}), \quad \forall \mathbf{V} = (\mathbf{v}, \mathbf{q}) \in \mathcal{H}^1, \quad (3.13)$$

where

$$B(\mathbf{U}, \mathbf{V}) = a_1(\mathbf{u}, \mathbf{v}) + a_2(p, q) + a_3(\mathbf{u}, q) + a_4(p, \mathbf{v}),$$

and

$$\begin{aligned}
a_1(\mathbf{u}, \mathbf{v}) &= \int_{\Omega} \left[\lambda (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v}) + \frac{\mu}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) : (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) - \rho \omega^2 \mathbf{u} \cdot \mathbf{v} \right] dx, \\
a_2(p, q) &= \int_{\Omega_R} (\nabla p \cdot \nabla \bar{q} - k^2 p \bar{q}) dx, \\
a_3(\mathbf{u}, q) &= \rho_f \omega^2 \int_{\Gamma} \mathbf{u} \cdot \mathbf{n} \bar{q} ds, \\
a_4(p, \mathbf{v}) &= \int_{\Gamma} \mathbf{n} p \cdot \bar{\mathbf{v}} ds, \\
b(p, q) &= - \int_{\Gamma_R} (Sp) \bar{q} ds
\end{aligned}$$

are sesquilinear forms defined on $(H^1(\Omega))^2 \times (H^1(\Omega))^2$, $H^1(\Omega_R) \times H^1(\Omega_R)$, $(H^1(\Omega))^2 \times H^1(\Omega_R)$, $H^1(\Omega_R) \times (H^1(\Omega))^2$ and $H^1(\Omega_R) \times H^1(\Omega_R)$, respectively, and ℓ defined by

$$\ell(\mathbf{V}) = \int_{\Gamma} \frac{\partial p^{inc}}{\partial \mathbf{n}} \bar{q} ds - \int_{\Gamma} \mathbf{n} p^{inc} \cdot \bar{\mathbf{v}} ds,$$

is a linear functional on \mathcal{H}^1 dependent on $(p^{inc}, \partial p^{inc} / \partial \mathbf{n}) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.

Remark 3.3. The double dot notation appeared in the above formulation is understood in the following way. If tensors \mathbf{A} and \mathbf{B} have rectangular Cartesian components a_{ij} and b_{ij} , $i, j = 1, 2$, respectively, then the double contraction of $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ is

$$\mathbf{A} : \mathbf{B} = \sum_{i=1}^2 \sum_{j=1}^2 a_{ij} b_{ij}.$$

Theorem 3.4. The sesquilinear form $A(\cdot, \cdot)$ in (3.13) satisfies

$$|A(\mathbf{U}, \mathbf{V})| \leq c \|\mathbf{U}\|_{\mathcal{H}^1} \|\mathbf{V}\|_{\mathcal{H}^1}, \quad \forall \mathbf{U}, \mathbf{V} \in \mathcal{H}^1.$$

where c is the continuity constant independent of \mathbf{U} and \mathbf{V} .

Proof. We begin with the sesquilinear form $a_1(\cdot, \cdot)$. The Cauchy-Schwarz inequality yields

$$\begin{aligned}
|a_1(\mathbf{u}, \mathbf{v})| &\leq (\lambda + 8\mu) \|\mathbf{u}\|_{(H^1(\Omega))^2} \|\mathbf{v}\|_{(H^1(\Omega))^2} + \rho \omega^2 \|\mathbf{u}\|_{(H^0(\Omega))^2} \|\mathbf{v}\|_{(H^0(\Omega))^2} \\
&\leq c \|\mathbf{u}\|_{(H^1(\Omega))^2} \|\mathbf{v}\|_{(H^1(\Omega))^2},
\end{aligned} \tag{3.14}$$

where $c > 0$ is a constant independent of \mathbf{u} and \mathbf{v} . Similarly, we have

$$|a_2(p, q)| \leq c \|p\|_{H^1(\Omega_R)} \|q\|_{H^1(\Omega_R)}, \tag{3.15}$$

where $c > 0$ is a constant independent of p and q . Also, according to the Cauchy-Schwarz inequality and the trace theorem, we know

$$\begin{aligned}
|a_3(\mathbf{u}, q)| &\leq c \|\mathbf{u}\|_{(H^{1/2}(\Gamma))^2} \|q\|_{H^{1/2}(\Gamma)} \\
&\leq c \|\mathbf{u}\|_{(H^1(\Omega))^2} \|q\|_{H^1(\Omega_R)}
\end{aligned} \tag{3.16}$$

and

$$|a_4(p, \mathbf{v})| \leq c \|p\|_{H^1(\Omega_R)} \|\mathbf{v}\|_{(H^1(\Omega))^2}, \tag{3.17}$$

where $c > 0$ in (3.16)-(3.17) are all constants independent of \mathbf{U} and \mathbf{V} . Next, we consider the sesquilinear form $b(\cdot, \cdot)$. The Cauchy-Schwarz inequality, the boundedness of the DtN mapping S and the trace theorem give

$$\begin{aligned} |b(p, q)| &\leq \|Sp\|_{H^{-1/2}(\Gamma_R)} \|q\|_{H^{1/2}(\Gamma_R)} \\ &\leq c \|p\|_{H^{1/2}(\Gamma_R)} \|q\|_{H^{1/2}(\Gamma_R)} \\ &\leq c \|p\|_{H^1(\Omega_R)} \|q\|_{H^1(\Omega_R)}, \end{aligned} \quad (3.18)$$

where $c > 0$ is a constant independent of p and q . Therefore, by combining (3.14)-(3.18) and applying the arithmetic-geometric mean inequality, we complete the proof. \square

Theorem 3.5. *The sesquilinear form $A(\cdot, \cdot)$ in (3.13) satisfies a Garding's inequality taking the form*

$$\operatorname{Re}\{A(\mathbf{V}, \mathbf{V})\} \geq \alpha \|\mathbf{V}\|_{\mathcal{H}^1}^2 - \beta \left(\|\mathbf{v}\|_{(H^{1/2+\varepsilon}(\Omega))^2}^2 + \|q\|_{H^{1/2+\varepsilon}(\Omega_R)}^2 \right), \quad \forall \mathbf{V} = (\mathbf{v}, q) \in \mathcal{H}^1.$$

where $\alpha > 0, \beta \geq 0$ and $0 < \varepsilon < 1/2$ are constants independent of \mathbf{V} .

Proof. We first consider the sesquilinear form $a_1(\cdot, \cdot)$. Suppose $\mathbf{v} = (v_1, v_2)$ and from the Korn's inequality ([23]) we know that there exists a constant $\alpha > 0$ such that

$$\sum_{i,j=1}^2 \|\epsilon_{ij}(\mathbf{v})\|_{H^0(\Omega)}^2 + \|\mathbf{v}\|_{(H^0(\Omega))^2}^2 \geq \alpha \|\mathbf{v}\|_{(H^1(\Omega))^2}^2, \quad (3.19)$$

where

$$\epsilon_{ij}(\mathbf{v}) = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \quad \forall 1 \leq i, j \leq 2.$$

If $\lambda \geq 0$, (3.19) yields

$$\begin{aligned} a_1(\mathbf{v}, \mathbf{v}) + \rho\omega^2 \|\mathbf{v}\|_{(H^0(\Omega))^2}^2 &= \lambda \int_{\Omega} |\nabla \cdot \mathbf{v}|^2 dx + \frac{\mu}{2} \int_{\Omega} (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) : (\nabla \bar{\mathbf{v}} + (\nabla \bar{\mathbf{v}})^T) dx \\ &\geq \frac{\mu}{2} \int_{\Omega} \left(4 \left| \frac{\partial v_1}{\partial x_1} \right|^2 + 2 \left| \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right|^2 + 4 \left| \frac{\partial v_2}{\partial x_2} \right|^2 \right) dx \\ &\geq 2\mu \left(\sum_{i,j=1}^2 \|\epsilon_{ij}(\mathbf{v})\|_{H^0(\Omega)}^2 + \|\mathbf{v}\|_{(H^0(\Omega))^2}^2 \right) - 2\mu \|\mathbf{v}\|_{(H^0(\Omega))^2}^2 \\ &\geq 2\mu\alpha \|\mathbf{v}\|_{(H^1(\Omega))^2}^2 - 2\mu \|\mathbf{v}\|_{(H^0(\Omega))^2}^2. \end{aligned} \quad (3.20)$$

If $\lambda < 0$, we have

$$\begin{aligned} a_1(\mathbf{v}, \mathbf{v}) + \rho\omega^2 \|\mathbf{v}\|_{(H^0(\Omega))^2}^2 &\geq 2(\lambda + \mu) \int_{\Omega} \left(\left| \frac{\partial v_1}{\partial x_1} \right|^2 + \left| \frac{\partial v_2}{\partial x_2} \right|^2 \right) dx + \mu \int_{\Omega} \left| \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right|^2 dx \\ &\geq 2(\lambda + \mu) \left(\sum_{i,j=1}^2 \|\epsilon_{ij}(\mathbf{v})\|_{H^0(\Omega)}^2 + \|\mathbf{v}\|_{(H^0(\Omega))^2}^2 \right) - 2(\lambda + \mu) \|\mathbf{v}\|_{(H^0(\Omega))^2}^2 \\ &\geq 2(\lambda + \mu)\alpha \|\mathbf{v}\|_{(H^1(\Omega))^2}^2 - 2(\lambda + \mu) \|\mathbf{v}\|_{(H^0(\Omega))^2}^2. \end{aligned} \quad (3.21)$$

Then a combination of (3.20) and (3.21) leads to

$$a_1(\mathbf{v}, \mathbf{v}) + \rho\omega^2 \|\mathbf{v}\|_{(H^0(\Omega))^2}^2 \geq 2(\lambda + \mu)\alpha \|\mathbf{v}\|_{(H^1(\Omega))^2}^2 - 2\mu \|\mathbf{v}\|_{(H^0(\Omega))^2}^2. \quad (3.22)$$

Therefore, the Sobolev embedding theorem gives

$$\operatorname{Re} \{a_1(\mathbf{v}, \mathbf{v})\} \geq \alpha \|\mathbf{v}\|_{(H^1(\Omega))^2}^2 - \beta \|\mathbf{v}\|_{(H^{1/2+\varepsilon}(\Omega))^2}^2, \quad (3.23)$$

where $\alpha > 0, \beta > 0$ and $0 < \varepsilon < 1/2$ are constants. Similarly, for the sesquilinear form $a_2(\cdot, \cdot)$, since

$$a_2(q, q) = \|q\|_{H^1(\Omega_R)}^2 - (k^2 + 1) \|q\|_{H^0(\Omega_R)}^2,$$

we conclude that

$$\operatorname{Re} \{a_2(q, q)\} \geq \|q\|_{H^1(\Omega_R)}^2 - \beta \|q\|_{H^{1/2+\varepsilon}(\Omega_R)}^2, \quad (3.24)$$

where $\beta > 0$ and $0 < \varepsilon < 1/2$ are constants. Next, we consider the sesquilinear form $a_3(\cdot, \cdot)$. The Hölder inequality and the arithmetic-geometric mean inequality give

$$\begin{aligned} |a_3(\mathbf{v}, q)| &\leq \rho_f \omega^2 \|\mathbf{v}\|_{(H^0(\Gamma))^2} \|q\|_{H^0(\Gamma)} \\ &\leq \beta \left(\|\mathbf{v}\|_{(H^0(\Gamma))^2}^2 + \|q\|_{H^0(\Gamma)}^2 \right). \end{aligned}$$

Then by the trace theorem and the Sobolev embedding theorem we have

$$\begin{aligned} \operatorname{Re} \{a_3(\mathbf{v}, q)\} &\geq -\beta \left(\|\mathbf{v}\|_{(H^0(\Gamma))^2}^2 + \|q\|_{H^0(\Gamma)}^2 \right) \\ &\geq -\beta \left(\|\mathbf{v}\|_{(H^{1/2}(\Omega))^2}^2 + \|q\|_{H^{1/2}(\Omega_R)}^2 \right) \\ &\geq -\beta \left(\|\mathbf{v}\|_{(H^{1/2+\varepsilon}(\Omega))^2}^2 + \|q\|_{H^{1/2+\varepsilon}(\Omega_R)}^2 \right), \end{aligned} \quad (3.25)$$

where $\beta > 0$ and $0 < \varepsilon < 1/2$ are constants. Similarly, for the sesquilinear form $a_4(\cdot, \cdot)$, it follows that

$$\operatorname{Re} \{a_4(q, \mathbf{v})\} \geq -\beta \left(\|\mathbf{v}\|_{(H^{1/2+\varepsilon}(\Omega))^2}^2 + \|q\|_{H^{1/2+\varepsilon}(\Omega_R)}^2 \right). \quad (3.26)$$

Finally, for the the sesquilinear form $b(\cdot, \cdot)$, we define

$$b_1(p, q) = \frac{1}{\pi} \sum_{n=1}^{\infty} n \int_0^{2\pi} \int_0^{2\pi} p(R, \phi) \overline{q(R, \theta)} \cos(n(\theta - \phi)) d\theta d\phi$$

and

$$b_2(p, q) = \frac{kR}{\pi} \sum_{n=0}^{\infty} \frac{H_n^{(n-1)}(kR)}{H_n^{(1)}(kR)} \int_0^{2\pi} \int_0^{2\pi} p(R, \phi) \overline{q(R, \theta)} \cos(n(\theta - \phi)) d\theta d\phi$$

which reads

$$b(p, q) = b_1(p, q) - b_2(p, q). \quad (3.27)$$

It follows from Theorem 4.2 in [27] that

$$b_1(q, q) \geq 0, \quad (3.28)$$

and

$$|\operatorname{Re} \{b_2(q, q)\}| \leq |b_2(q, q)| \leq c \|q\|_{H^0(\Gamma_R)}^2. \quad (3.29)$$

A combination of (3.27)-(3.29) implies that

$$\operatorname{Re} \{b(q, q)\} \geq -\beta \|q\|_{H^{1/2+\varepsilon}(\Omega_R)}^2, \quad (3.30)$$

because of the Sobolev embedding theorem where $\beta > 0$ and $0 < \varepsilon < 1/2$ are constants. Therefore, a combination of (3.23)-(3.26) and (3.30) yields (3.19). This completes the proof. \square

Now, the existence result follows immediately from the Fredholm Alternative theorem: Uniqueness implies the existence. As a consequence of Theorem 3.4 and 3.5, we have the following theorem.

Theorem 3.6. *Let the surface Γ and the material parameter (μ, λ, ρ) be such that there are no traction free solutions, then the variational equation (3.13) admits a unique solution.*

4 Modified nonlocal boundary value problem

4.1 Truncated DtN mapping

In practical computing, one needs to truncate the infinite series of the exact DtN mapping at a finite order to obtain an approximate DtN mapping written as

$$S^N \varphi := \sum_{n=0}^N \frac{k H_n^{(1)'}(kR)}{\pi H_n^{(1)}(kR)} \int_0^{2\pi} \varphi(R, \phi) \cos(n(\theta - \phi)) d\phi, \quad (4.1)$$

or

$$S^N \varphi := \sum_{n=-N}^N \frac{k H_n^{(1)'}(kR)}{2\pi H_n^{(1)}(kR)} \int_0^{2\pi} \varphi(R, \phi) e^{in(\theta - \phi)} d\phi, \quad (4.2)$$

for all $\varphi \in H^s(\Gamma_R)$, $s \geq 1/2$. Obviously, S^N is also a linear bounded operator. Here, the non-negative integer N is called the truncation order of the DtN mapping. Consequently, we arrive at a truncated nonlocal boundary value problem consisting of (3.5)-(3.8) and

$$\frac{\partial p}{\partial \mathbf{n}} = S^N p \quad \text{on } \Gamma_R. \quad (4.3)$$

We include the point estimate for the difference of S and S^N proposed in [27] in the next theorem.

Theorem 4.1. *Suppose that the DtN mappings S and S^N are defined as in (3.2) and (4.1), respectively. Then, for given $\varphi \in H^s(\Gamma_R)$, $s \in \mathbb{R}$, there holds, for all $t \geq 0$,*

$$\|(S - S^N)\varphi\|_{H^{s-1}(\Gamma_R)} \leq c \frac{\epsilon(N, \varphi)}{N^t} \|\varphi\|_{H^{s+t}(\Gamma_R)}, \quad (4.4)$$

where $c > 0$ is a constant independent of φ and N , $\epsilon(N, \varphi)$ is a function of the truncation order N and φ for given values of s and t , generated by the addition of leading terms to the summation with positive terms for the construction of the norm on the space $H^{s+t}(\Gamma_R)$, satisfying $\epsilon(N, \varphi) \leq 1$ and $\epsilon(N, \varphi) \rightarrow 0$ as $N \rightarrow \infty$ for all $\varphi \in H^s(\Gamma_R)$.

From the estimation (4.4) we can observe that the truncation error between S and S^N tends to zero as $N \rightarrow \infty$. However, when $t = 0$, the attenuation property depends on $\epsilon(N, \varphi)$ and we know few about this function. We now give a new and more apparent truncation error estimate for some special $\varphi \in H^s(\Gamma_R)$ which will be used in the subsequent discussions.

Theorem 4.2. *Suppose that the DtN mappings S and S^N are defined as in (3.3) and (4.2), respectively. Let p be a solution of Helmholtz equation outside Ω satisfying either (3.1) or (4.3). Then there exists a $N_0 > 0$ such that for all $N > N_0$,*

$$\|(S - S^N)p\|_{H^{s-1}(\Gamma_R)} \leq cq^N \|p\|_{H^{s+t+1/2}(\Omega_R)}, \quad \forall t \geq 0, s \geq 1/2, \quad (4.5)$$

where $0 < q < 1$ is a constant independent of N .

Proof. Case 1: Suppose that p satisfies (3.1).

We know that p admits an expansion taking the form

$$p(x) = \sum_{n \in \mathbb{Z}} p_n \frac{H_n^{(1)}(k|x|)}{H_n^{(1)}(k|R|)} e^{in\theta} \quad \text{for all } |x| > \Gamma^+.$$

Here, $\Gamma^+ := \max\{|x|, x \in \Gamma\}$. It easily follows that

$$\begin{aligned}
\|(S - S^N)p\|_{H^{s-1}(\Gamma_R)} &= \left\{ \sum_{|n|>N} (1+n^2)^s |p_n|^2 \frac{k^2}{1+n^2} \left| \frac{H_n^{(1)'}(kR)}{H_n^{(1)}(kR)} \right|^2 \right\}^{1/2} \\
&\leq c \left\{ \sum_{|n|>N} (1+n^2)^s |p_n|^2 \right\}^{1/2} \\
&\leq c \sum_{|n|>N} (1+n^2)^{s/2} |p_n|.
\end{aligned}$$

For some small $\varepsilon > 0$, we have

$$\begin{aligned}
(1+n^2)^{s/2} |p_n| \left| \frac{H_n^{(1)}(k(\Gamma^+ + 2\varepsilon))}{H_n^{(1)}(kR)} \right| &= (1+n^2)^{(s+t)/2} |p_n| \left| \frac{H_n^{(1)}(k(\Gamma^+ + \varepsilon))}{H_n^{(1)}(kR)} \right| \\
&\times (1+n^2)^{-t/2} \left| \frac{H_n^{(1)}(k(\Gamma^+ + 2\varepsilon))}{H_n^{(1)}(k(\Gamma^+ + \varepsilon))} \right| \\
&\leq \left\{ \sum_{n \in \mathbb{Z}} (1+n^2)^{s+t} |p_n|^2 \left| \frac{H_n^{(1)}(k(\Gamma^+ + \varepsilon))}{H_n^{(1)}(kR)} \right|^2 \right\}^{1/2} \\
&\times \left\{ \sum_{n \in \mathbb{Z}} (1+n^2)^{-t} \left| \frac{H_n^{(1)}(k(\Gamma^+ + 2\varepsilon))}{H_n^{(1)}(k(\Gamma^+ + \varepsilon))} \right|^2 \right\}^{1/2}
\end{aligned}$$

Note that for sufficiently large $|n|$ and $z > 0$,

$$H_{|n|}^{(1)}(z) \sim -i \sqrt{\frac{2}{|n|\pi}} \left(\frac{e}{2|n|} \right)^{-|n|} z^{-|n|}.$$

Then we deduce that there exists a $N_0 > 0$ such that for $|n| > N_0$,

$$(1+n^2)^{-t} \left| \frac{H_n^{(1)}(k(\Gamma^+ + 2\varepsilon))}{H_n^{(1)}(k(\Gamma^+ + \varepsilon))} \right|^2 \leq c(1+n^2)^{-t} \left(\frac{\Gamma^+ + \varepsilon}{\Gamma^+ + 2\varepsilon} \right)^n,$$

and

$$\left| \frac{H_n^{(1)}(kR)}{H_n^{(1)}(k(\Gamma^+ + 2\varepsilon))} \right| \leq cq^{|n|},$$

where $q := (\Gamma^+ + 2\varepsilon)/R < 1$. These further imply that

$$\sum_{n \in \mathbb{Z}} (1+n^2)^{-t} \left| \frac{H_n^{(1)}(k(\Gamma^+ + 2\varepsilon))}{H_n^{(1)}(k(\Gamma^+ + \varepsilon))} \right|^2 < \infty.$$

Then for $|n| > N_0$, we deduce from the trace theorem that

$$(1+n^2)^{s/2} |p_n| \leq cq^{|n|} \|p\|_{H^{s+t}(\Gamma_{\Gamma^++\varepsilon})} \leq cq^{|n|} \|p\|_{H^{s+t+1/2}(\Omega_R)}.$$

We conclude that

$$\begin{aligned} \|(S - S^N)p\|_{H^{s-1}(\Gamma_R)} &\leq c \sum_{|n|>N} q^{|n|} \|p\|_{H^{s+t+1/2}(\Omega_R)} \\ &\leq cq^N \|p\|_{H^{s+t+1/2}(\Omega_R)}. \end{aligned}$$

Case 2: Suppose that p satisfies (4.3).

We know that p admits an expansion taking the form

$$p(x) = \sum_{n \in \mathbb{Z}} \left(p_n^{(1)} \frac{H_n^{(1)}(k|x|)}{H_n^{(1)}(k|R|)} + p_n^{(2)} \frac{H_n^{(2)}(k|x|)}{H_n^{(2)}(k|R|)} \right) e^{in\theta} \quad \text{for all } |x| > \Gamma^+.$$

According to the condition (4.3), we conclude that

$$p_n^{(1)} \frac{H_n^{(1)'}(kR)}{H_n^{(1)}(kR)} + p_n^{(2)} \frac{H_n^{(2)'}(kR)}{H_n^{(2)}(kR)} = \begin{cases} (p_n^{(1)} + p_n^{(2)}) \frac{H_n^{(1)'}(kR)}{H_n^{(1)}(kR)}, & |n| \leq N, \\ 0, & |n| > N, \end{cases}$$

which implies that

$$p_n^{(2)} = \begin{cases} 0, & |n| \leq N, \\ -p_n^{(1)} \frac{H_n^{(1)'}(kR)}{H_n^{(1)}(kR)} \frac{H_n^{(2)}(kR)}{H_n^{(2)'}(kR)}, & |n| > N, \end{cases}$$

It follows that

$$\begin{aligned} \|(S - S^N)p\|_{H^{s-1}(\Gamma_R)} &= \left\{ \sum_{|n|>N} (1+n^2)^s |p_n^{(1)} + p_n^{(2)}|^2 \frac{k^2}{1+n^2} \left| \frac{H_n^{(1)'}(kR)}{H_n^{(1)}(kR)} \right|^2 \right\}^{1/2} \\ &\leq c \left\{ \sum_{|n|>N} (1+n^2)^s |p_n^{(1)} + p_n^{(2)}|^2 \right\}^{1/2} \\ &\leq c \sum_{|n|>N} (1+n^2)^{s/2} |p_n^{(1)}| \left| 1 - \frac{H_n^{(1)'}(kR)}{H_n^{(1)}(kR)} \frac{H_n^{(2)}(kR)}{H_n^{(2)'}(kR)} \right|. \end{aligned}$$

The proof of the remaining assertions is analogous to that of Case 1. \square

4.2 Modified weak formulation

We now consider the modified variational equation of (3.13) due to the truncation of the DtN mapping: looking for $\mathbf{U}_N = (\mathbf{u}_N, p_N) \in \mathcal{H}^1$ such that

$$A^N(\mathbf{U}_N, \mathbf{V}) = B(\mathbf{U}_N, \mathbf{V}) + b^N(p_N, q) = \ell(\mathbf{V}), \quad \forall \mathbf{V} = (\mathbf{v}, q) \in \mathcal{H}^1, \quad (4.6)$$

where

$$b^N(p_N, q) = - \int_{\Gamma_R} (S^N p_N) \bar{q} ds.$$

Theorem 4.3. *The sesquilinear form $A^N(\cdot, \cdot)$ in (4.6) satisfies*

$$1. \quad |A^N(\mathbf{U}, \mathbf{V})| \leq c \|\mathbf{U}\|_{\mathcal{H}^1} \|\mathbf{V}\|_{\mathcal{H}^1}, \quad \forall \mathbf{U}, \mathbf{V} \in \mathcal{H}^1, \quad (4.7)$$

$$2. \quad \operatorname{Re} \{A^N(\mathbf{V}, \mathbf{V})\} \geq \alpha \|\mathbf{V}\|_{\mathcal{H}^1}^2 - \beta \left(\|\mathbf{v}\|_{(H^{1/2+\varepsilon}(\Omega))^2}^2 + \|q\|_{H^{1/2+\varepsilon}(\Omega_R)}^2 \right), \quad \forall \mathbf{V} = (\mathbf{v}, q) \in \mathcal{H}^1, \quad (4.8)$$

where $c > 0, \alpha > 0, \beta \geq 0$ and $0 < \varepsilon < 1/2$ are constants independent of \mathbf{U} and \mathbf{V} .

Proof. It is easy to deduce (4.7) due to the fact that S^N is also bounded. We now show the proof of (4.8). Following (3.27), we define

$$b_1^N(p, q) = \frac{1}{\pi} \sum_{n=1}^N n \int_0^{2\pi} \int_0^{2\pi} p(R, \phi) \overline{q(R, \theta)} \cos(n(\theta - \phi)) d\theta d\phi$$

and

$$b_2^N(p, q) = \frac{kR}{\pi} \sum_{n=0}^N \frac{H_{n-1}^{(1)}(kR)}{H_n^{(1)}(kR)} \int_0^{2\pi} \int_0^{2\pi} p(R, \phi) \overline{q(R, \theta)} \cos(n(\theta - \phi)) d\theta d\phi$$

which yields

$$b^N(p, q) = b_1^N(p, q) - b_2^N(p, q).$$

It follows from Theorem 4.4 in [27] that

$$b_1^N(q, q) \geq 0,$$

and

$$|\operatorname{Re} \{b_2^N(q, q)\}| \leq |b_2^N(q, q)| \leq c \|q\|_{H^0(\Gamma_R)}^2$$

and these further imply that

$$\operatorname{Re} \{b^N(q, q)\} \geq -\beta \|q\|_{H^{1/2+\varepsilon}(\Omega_R)}^2, \quad (4.9)$$

where $\beta > 0$ and $0 < \varepsilon < 1/2$ are constants. Consequently, by the combination of (3.23)-(3.26) and (4.9), we completes the proof of (4.8). \square

Now, we present some preliminary results before showing the uniqueness (Theorem 4.11) of the modified variational equation (4.6) which is the key ingredient of this work.

Definition 4.4. Let $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ be inner products on $(H^1(\Omega))^2 \times (H^1(\Omega))^2$ and $H^1(\Omega_R) \times H^1(\Omega_R)$ defined by, for $\forall \mathbf{u}, \mathbf{v} \in (H^1(\Omega))^2$ and $\forall p, q \in H^1(\Omega_R)$,

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v} \rangle_1 &= \int_{\Omega} \left[\lambda (\nabla \cdot \mathbf{u}) (\nabla \cdot \mathbf{v}) + \frac{\mu}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) : (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) + 2\mu \mathbf{u} \cdot \mathbf{v} \right] dx, \\ \langle p, q \rangle_2 &= \int_{\Omega_R} (\nabla p \cdot \nabla \bar{q} + p \bar{q}) dx + b_1(p, q), \end{aligned}$$

which further induce the norms $||| \cdot |||_1$ on $(H^1(\Omega))^2$ and $||| \cdot |||_2$ on $H^1(\Omega_R)$, respectively, i.e., $\forall \mathbf{u} \in (H^1(\Omega))^2, p \in H^1(\Omega_R)$,

$$|||\mathbf{u}|||_1^2 = \langle \mathbf{u}, \mathbf{u} \rangle_1, \quad |||p|||_2^2 = \langle p, p \rangle_2.$$

From the above definition together with (3.22) and (3.28), we conclude that there exist constants $\alpha_0 > 0, \beta_0 > 0, \gamma_0 > 0$ such that

$$\alpha_0 |||\mathbf{u}|||_{(H^1(\Omega))^2}^2 \leq |||\mathbf{u}|||_1^2 \leq \beta_0 |||\mathbf{u}|||_{(H^1(\Omega))^2}^2, \quad (4.10)$$

$$|||p|||_{H^1(\Omega_R)}^2 \leq |||p|||_2^2 \leq \gamma_0 |||p|||_{H^1(\Omega_R)}^2. \quad (4.11)$$

Now, let $\{\tilde{\Phi}_i, \tilde{\lambda}_i\}$ and $\{\hat{\varphi}_i, \hat{\lambda}_i\}$ be eigenpairs satisfying

$$\langle \tilde{\Phi}_i, \Theta \rangle_1 = \tilde{\lambda}_i \left(\tilde{\Phi}_i, \Theta \right)_{(L^2(\Omega))^2}, \quad \forall \Theta \in (H^1(\Omega))^2, \quad (4.12)$$

$$\langle \hat{\varphi}_i, \theta \rangle_2 = \hat{\lambda}_i (\hat{\varphi}_i, \theta)_{L^2(\Omega_R)}, \quad \forall \theta \in H^1(\Omega_R), \quad (4.13)$$

where $(\cdot, \cdot)_H$ is the classical L^2 inner product on H . Without loss of generalities, we assume that $0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots$, $0 < \hat{\lambda}_1 \leq \hat{\lambda}_2 \leq \dots$, and $(\tilde{\Phi}_i, \tilde{\Phi}_j)_{(L^2(\Omega))^2} = (\hat{\varphi}_i, \hat{\varphi}_j)_{L^2(\Omega_R)} = \delta_{ij}$.

Lemma 4.5. Let $\mathbf{u} = \sum_{n=1}^{\infty} \tilde{c}_n \tilde{\Phi}_n$, $p = \sum_{n=1}^{\infty} \hat{c}_n \hat{\varphi}_n$ and $\mathbf{U} = (\mathbf{u}, p)$.
(1). If $\mathbf{U} \in \mathcal{H}^0$,

$$\|\mathbf{U}\|_{\mathcal{H}^0}^2 = \sum_{n=1}^{\infty} (|\tilde{c}_n|^2 + |\hat{c}_n|^2). \quad (4.14)$$

(2). If $\mathbf{U} \in \mathcal{H}^1$,

$$\|\mathbf{u}\|_1^2 = \sum_{n=1}^{\infty} \tilde{\lambda}_i |\tilde{c}_n|^2, \quad \|p\|_2^2 = \sum_{n=1}^{\infty} \hat{\lambda}_i |\hat{c}_n|^2. \quad (4.15)$$

(3). Let $H_{M_1} = \text{span}_{\tilde{\lambda}_i \leq M_1} \{\tilde{\Phi}_i\}$, $H_{M_2} = \text{span}_{\hat{\lambda}_i \leq M_2} \{\hat{\varphi}_i\}$ and we define

$$\begin{aligned} H_{M_1}^\perp &= \{\mathbf{v} \in (H^1(\Omega))^2 : \langle \mathbf{v}, \Theta \rangle_1 = 0, \quad \forall \Theta \in H_{M_1}\}, \\ H_{M_2}^\perp &= \{q \in H^1(\Omega_R) : \langle q, \theta \rangle_1 = 0, \quad \forall \theta \in H_{M_2}\}. \end{aligned}$$

Then we have

$$\|\mathbf{u}\|_1^2 \leq M_1 \|\mathbf{u}\|_{(H^0(\Omega))^2}^2, \quad \|p\|_2^2 \leq M_2 \|p\|_{H^0(\Omega_R)}^2, \quad \forall \mathbf{U} \in H_{M_1} \times H_{M_2}, \quad (4.16)$$

and

$$\|\mathbf{u}\|_{(H^0(\Omega))^2}^2 \leq \frac{1}{M_1} \|\mathbf{u}\|_1^2, \quad \|p\|_{H^0(\Omega_R)}^2 \leq \frac{1}{M_2} \|p\|_2^2, \quad \forall \mathbf{U} \in H_{M_1}^\perp \times H_{M_2}^\perp. \quad (4.17)$$

Lemma 4.6. Suppose $\mathbf{U} = (\mathbf{u}, p) \in \mathcal{H}^1$ satisfying

$$\begin{aligned} \langle \mathbf{u}, \Theta \rangle_1 &= \langle \tilde{\mathbf{f}}_1, \Theta \rangle, \quad \forall \Theta \in (H^1(\Omega))^2, \\ \langle p, \theta \rangle_2 &= \langle \hat{f}_2, \theta \rangle, \quad \forall \theta \in H^1(\Omega_R) \end{aligned}$$

with $(\tilde{\mathbf{f}}_1, \hat{f}_2) \in \mathcal{H}^0$. Then there exists a positive constant c such that

$$\|\mathbf{u}\|_{(H^2(\Omega))^2} \leq c \|\tilde{\mathbf{f}}_1\|_{(H^0(\Omega))^2}, \quad \|p\|_{H^2(\Omega_R)} \leq c \|\hat{f}_2\|_{H^0(\Omega_R)}. \quad (4.18)$$

Proof. The first assertion for \mathbf{u} follows from the interior regularity estimates ([11]). We now prove the regularity results for p . It follows that $b_1(\cdot, \cdot)$ is the corresponding sesquilinear form of the DtN mapping for the solution of homogeneous Laplace equation outside Ω . Now we define

$$\tilde{p}(x) = \begin{cases} p(x), & |x| \leq R, \\ \sum_{n \geq 0} \left(\frac{|x|}{R}\right)^n (a_n \cos n\theta + b_n \sin n\theta), & |x| > R, \end{cases}$$

where $p(R, \theta) = \sum_{n \geq 0} (a_n \cos n\theta + b_n \sin n\theta)$. Note that for $\varphi \in C_0^\infty(\mathbb{R}^2)$,

$$\begin{aligned} \int_{\mathbb{R}^2} \nabla \tilde{p} \cdot \nabla \bar{\varphi} dx + \int_{\Omega_R} \tilde{p} \bar{\varphi} dx &= \langle \tilde{p}, \varphi \rangle_2 - \int_{|x| \geq R} \Delta \tilde{p} \bar{\varphi} dx \\ &= \langle p, \varphi \rangle_2 \\ &= \langle \hat{f}_2, \varphi \rangle. \end{aligned}$$

Then we conclude from interior regularity estimates that $\tilde{p} = p \in H^2(\Omega_R)$ and

$$\begin{aligned} \|p\|_{H^2(\Omega_R)} &\leq c \{\|\tilde{p}\|_{H^1(\Omega_{R+1})} + \|\hat{f}_2\|_{H^0(\Omega_R)}\} \\ &\leq c \|\hat{f}_2\|_{H^0(\Omega_R)}, \end{aligned}$$

in which the last inequality follows from the fact that \tilde{p} satisfies a stable variational problem on $H^1(\Omega_{R+1})$. \square

Corollary 4.7. If $\mathbf{U} = (\mathbf{u}, p) \in H_{M_1} \times H_{M_2}$,

$$\|\mathbf{u}\|_{(H^2(\Omega))^2} \leq cM_1^{1/2} \|\mathbf{u}\|_1, \quad \|p\|_{H^2(\Omega_R)} \leq cM_2^{1/2} \|p\|_2. \quad (4.19)$$

Proof. For $\mathbf{U} = (\mathbf{u}, p) \in H_{M_1} \times H_{M_2}$, we can see that

$$\mathbf{u} = \sum_{\tilde{\lambda}_n \leq M_1} \tilde{c}_n \tilde{\Phi}_n, \quad p = \sum_{\hat{\lambda}_n \leq M_2} \hat{c}_n \hat{\varphi}_n.$$

Then (4.12) and (4.13) imply that

$$\begin{aligned} \langle \mathbf{u}, \Theta \rangle_1 &= \left(\sum_{\tilde{\lambda}_n \leq M_1} \tilde{\lambda}_n \tilde{c}_n \tilde{\Phi}_n, \Theta \right), \quad \forall \Theta \in (H^1(\Omega))^2, \\ \langle p, \theta \rangle_2 &= \left(\sum_{\hat{\lambda}_n \leq M_2} \hat{\lambda}_n \hat{c}_n \hat{\varphi}_n, \theta \right), \quad \forall \theta \in H^1(\Omega_R). \end{aligned}$$

Therefore, (4.19) follows from Lemma 4.6 and (4.15). \square

Corollary 4.8. If $\mathbf{U} = (\mathbf{u}, p) \in H_{M_1}^\perp \times H_{M_2}^\perp$, there exist positive constants M_0, C_1 such that for $M_1, M_2 \geq M_0$,

$$\|\mathbf{U}\|_{\mathcal{H}^1}^2 \leq C_1 \operatorname{Re}\{A^N(\mathbf{U}, \mathbf{U})\}. \quad (4.20)$$

Proof. From (4.10) and (4.11), we know

$$\alpha_0 \|\mathbf{U}\|_{\mathcal{H}^1}^2 \leq \|\mathbf{u}\|_1^2 + \alpha_0 \|p\|_2^2 \leq \max\{\alpha_0, 1\} (\|\mathbf{u}\|_1^2 + \|p\|_2^2).$$

According to Definition 4.1 and $b_1^N(p, p) \geq 0$, we have

$$\begin{aligned} \alpha_0 \|\mathbf{U}\|_{\mathcal{H}^1}^2 &\leq \max\{\alpha_0, 1\} \operatorname{Re}\{A^N(\mathbf{U}, \mathbf{U})\} + \max\{\alpha_0, 1\} (2\mu + \rho\omega^2) \|\mathbf{u}\|_{(H^0(\Omega))^2}^2 + \max\{\alpha_0, 1\} (k^2 + 1) \|p\|_{H^0(\Omega_R)}^2 \\ &\quad + \max\{\alpha_0, 1\} \operatorname{Re}\{b_2^N(p, p) - a_3(\mathbf{u}, p) - a_4(p, \mathbf{u})\}. \end{aligned} \quad (4.21)$$

The Sobolev embedding theorem and the interpolation inequality for $0 < \theta = 1/2 + \varepsilon < 1$, $0 < \varepsilon < 1/2$ gives

$$\begin{aligned} |b_2^N(p, p) - a_3(\mathbf{u}, p) - a_4(p, \mathbf{u})| &\leq c(\|\mathbf{u}\|_{(H^0(\Gamma))^2}^2 + \|p\|_{H^0(\Gamma_R)}^2) \\ &\leq c(\|\mathbf{u}\|_{(H^{1/2}(\Omega))^2}^2 + \|p\|_{H^{1/2}(\Omega_R)}^2) \\ &\leq c(\|\mathbf{u}\|_{(H^{1/2+\varepsilon}(\Omega))^2}^2 + \|p\|_{H^{1/2+\varepsilon}(\Omega_R)}^2) \\ &\leq c(\|\mathbf{u}\|_{(H^0(\Omega))^2}^{1-2\varepsilon} \|\mathbf{u}\|_{(H^1(\Omega))^2}^{1+2\varepsilon} + \|p\|_{H^0(\Omega_R)}^{1-2\varepsilon} \|p\|_{H^1(\Omega_R)}^{1+2\varepsilon}). \end{aligned} \quad (4.22)$$

A combination of (4.10), (4.11) and (4.17) leads to

$$\|\mathbf{u}\|_{(H^0(\Omega))^2}^2 \leq \frac{\beta_0}{M_1} \|\mathbf{u}\|_{(H^1(\Omega))^2}^2, \quad \|p\|_{H^0(\Omega_R)}^2 \leq \frac{\gamma_0}{M_2} \|p\|_{H^1(\Omega_R)}^2. \quad (4.23)$$

Thus, we conclude from (4.21)-(4.23) that

$$\begin{aligned} \|\mathbf{U}\|_{\mathcal{H}^1}^2 &\leq C_\alpha \operatorname{Re}\{A^N(\mathbf{U}, \mathbf{U})\} + C_\alpha (2\mu + \rho\omega^2) \|\mathbf{u}\|_{(H^0(\Omega))^2}^2 + C_\alpha (k^2 + 1) \|p\|_{H^0(\Omega_R)}^2 \\ &\quad + C_\alpha |b_2^N(p, p) - a_3(\mathbf{u}, p) - a_4(p, \mathbf{u})| \\ &\leq C_\alpha \operatorname{Re}\{A^N(\mathbf{U}, \mathbf{U})\} \\ &\quad + C_\alpha \frac{\beta_0^2 (2\mu + \rho\omega^2)}{M_1^2} \|\mathbf{u}\|_{(H^1(\Omega))^2}^2 + C_\alpha \frac{\gamma_0^2 (1 + k^2)}{M_2^2} \|p\|_{H^1(\Omega_R)}^2 \\ &\quad + C_\alpha \frac{c\beta_0^{1/2-\varepsilon}}{M_1^{1/2-\varepsilon}} \|\mathbf{u}\|_{(H^1(\Omega))^2}^2 + C_\alpha \frac{c\gamma_0^{1/2-\varepsilon}}{M_2^{1/2-\varepsilon}} \|p\|_{H^1(\Omega_R)}^2, \end{aligned} \quad (4.24)$$

where $C_\alpha = \max\{1, 1/\alpha_0\}$. Let $M_0 > 0$ be such that

$$1 - C_\alpha \frac{\beta_0^2(2\mu + \rho\omega^2)}{M_0^2} - C_\alpha \frac{c\beta_0^{1/2-\varepsilon}}{M_0^{1/2-\varepsilon}} > \frac{1}{2}, \quad (4.25)$$

$$1 - C_\alpha \frac{\gamma_0^2(1+k^2)}{M_0^2} - C_\alpha \frac{c\gamma_0^{1/2-\varepsilon}}{M_0^{1/2-\varepsilon}} > \frac{1}{2}. \quad (4.26)$$

Then for $M_1, M_2 \geq M_0$, there exists a constant $C_1 = 2C_\alpha$ such that (4.20) holds. \square

Corollary 4.9. *If $\mathbf{U} = (\mathbf{u}, p) \in H_{M_1} \times H_{M_2}$ and $\mathbf{V} = (\mathbf{v}, q) \in H_{M_1}^\perp \times H_{M_2}^\perp$, we have*

$$|A(\mathbf{U}, \mathbf{V})| \leq c\gamma_0^{1/4-\varepsilon/2} M_2^{\varepsilon/2-1/4} \|p\|_{H^1(\Omega_R)} \|q\|_{H^1(\Omega_R)}. \quad (4.27)$$

where $c > 0$ and $0 < \varepsilon < 1/2$ are constants independent of \mathbf{U} and \mathbf{V} .

Proof. It easily follows that

$$\begin{aligned} |A(\mathbf{U}, \mathbf{V})| &= |\langle \mathbf{u}, \mathbf{v} \rangle_1 - (2\mu + \rho\omega^2)(\mathbf{u}, \mathbf{v})_{(H^0(\Omega))^2} + \langle p, q \rangle_2 - (k^2 + 1)(p, q)_{H^0(\Omega_R)} - b_2(p, q)| \\ &= |b_2(p, q)| \\ &\leq c\|p\|_{H^{1/2+\varepsilon}(\Omega_R)} \|q\|_{H^{1/2+\varepsilon}(\Omega_R)}, \end{aligned}$$

where $c > 0$ is a constant. Then the interpolation inequality and (4.23) leads to (4.27). \square

Corollary 4.10. *If $\mathbf{U} = (\mathbf{u}, p) \in H_{M_1} \times H_{M_2}$, there exist $\mathbf{V} = (\mathbf{v}, q) \in H_{M_1} \times H_{M_2}$ and a positive constant c such that*

$$\|\mathbf{U}\|_{\mathcal{H}^1} \leq c \frac{\text{Re}\{A(\mathbf{U}, \mathbf{V})\}}{\|\mathbf{V}\|_{\mathcal{H}^1}}. \quad (4.28)$$

Proof. Firstly, we observe that for $\mathbf{W} = (\mathbf{w}, \varpi) \in \mathcal{H}^1$, we can rewrite it as

$$\mathbf{W} = \mathbf{W}_M + \mathbf{W}_M^\perp = (\mathbf{w}_{M_1} + \mathbf{w}_{M_1}^\perp, \varpi_{M_2} + \varpi_{M_2}^\perp), \quad (4.29)$$

where $\mathbf{w}_{M_1} \in H_{M_1}$, $\mathbf{w}_{M_1}^\perp \in H_{M_1}^\perp$, $\varpi_{M_2} \in H_{M_2}$ and $\varpi_{M_2}^\perp \in H_{M_2}^\perp$. Then from (4.10) and (4.11) we know that

$$\begin{aligned} \|\mathbf{W}_M\|_{\mathcal{H}^1}^2 &= \|\mathbf{w}_{M_1}\|_{(H^1(\Omega))^2}^2 + \|\varpi_{M_2}\|_{H^1(\Omega_R)}^2 \\ &\leq \frac{1}{\alpha_0} \|\mathbf{w}_{M_1}\|_1 + \|\varpi_{M_2}\|_2 \\ &\leq \frac{1}{\alpha_0} \|\mathbf{w}\|_1 + \|\varpi\|_2 \\ &\leq \frac{\beta_0}{\alpha_0} \|\mathbf{w}\|_{(H^1(\Omega))^2}^2 + \gamma_0 \|\varpi\|_{H^1(\Omega_R)}^2 \\ &\leq \max\left\{\frac{\beta_0}{\alpha_0}, \gamma_0\right\} \|\mathbf{W}\|_{\mathcal{H}^1}^2. \end{aligned} \quad (4.30)$$

Similarly, we have

$$\|\mathbf{W}_M^\perp\|_{\mathcal{H}^1}^2 \leq \max\left\{\frac{\beta_0}{\alpha_0}, \gamma_0\right\} \|\mathbf{W}\|_{\mathcal{H}^1}^2. \quad (4.31)$$

From the stability of (3.13), we know that there exists a $\mathbf{W} = (\mathbf{w}, \varpi) \in \mathcal{H}^1$ such that

$$\|\mathbf{U}\|_{\mathcal{H}^1} \leq c \frac{\text{Re}\{A(\mathbf{U}, \mathbf{W})\}}{\|\mathbf{W}\|_{\mathcal{H}^1}},$$

where $c > 0$ is a constant. Let $\mathbf{V} = \mathbf{W}_M$. Therefore, Corollary 4.3, (4.30) and (4.31) yield

$$\begin{aligned}\|\mathbf{U}\|_{\mathcal{H}^1} &\leq c \frac{\operatorname{Re}\{A(\mathbf{U}, \mathbf{W})\}}{\|\mathbf{W}\|_{\mathcal{H}^1}} \\ &= c \frac{\operatorname{Re}\{A(\mathbf{U}, \mathbf{V}) + A(\mathbf{U}, \mathbf{W} - \mathbf{V})\}}{\|\mathbf{W}\|_{\mathcal{H}^1}} \\ &\leq c \max\left\{\frac{\beta_0}{\alpha_0}, \gamma_0\right\} \frac{\operatorname{Re}\{A(\mathbf{U}, \mathbf{V})\}}{\|\mathbf{V}\|_{\mathcal{H}^1}} + c\gamma_0^{1/4-\varepsilon/2} M_2^{\varepsilon/2-1/4} \max\left\{\frac{\beta_0}{\alpha_0}, \gamma_0\right\} \|\mathbf{U}\|_{\mathcal{H}^1}.\end{aligned}$$

Let $M_0 > 0$ be such that

$$1 - c\gamma_0^{1/4-\varepsilon/2} M_0^{\varepsilon/2-1/4} \max\left\{\frac{\beta_0}{\alpha_0}, \gamma_0\right\} > \frac{1}{2}. \quad (4.32)$$

Thus, (4.28) holds for $M_2 \geq M_0$. \square

Theorem 4.11. *Let the surface Γ and the material parameter (μ, λ, ρ) be such that there are no traction free solutions, then there exists a constant $N_0 = N_0 \geq 0$ such that the modified variational equation (4.6) has at most one solution for $N \geq N_0$.*

Proof. Here we prove that for all $\mathbf{U} = (\mathbf{u}, p) \in \mathcal{H}^1$, there exists a constant $N_0 = N_0 \geq 0$ such that

$$\|\mathbf{U}\|_{\mathcal{H}^1} \leq c \sup_{(\mathbf{0}, 0) \neq \mathbf{V} \in \mathcal{H}_h} \frac{|A^N(\mathbf{U}, \mathbf{V})|}{\|\mathbf{V}\|_{\mathcal{H}^1}}, \quad (4.33)$$

where $c > 0$ is a constant. Then the Theorem follows immediately. We use the same form as (4.29) to represent \mathbf{U} as

$$\mathbf{U} = \mathbf{U}_M + \mathbf{U}_M^\perp = (\mathbf{u}_{M_1} + \mathbf{u}_{M_1}^\perp, p_{M_2} + p_{M_2}^\perp),$$

where $\mathbf{u}_{M_1} \in H_{M_1}$, $\mathbf{u}_{M_1}^\perp \in H_{M_1}^\perp$, $p_{M_2} \in H_{M_2}$ and $p_{M_2}^\perp \in H_{M_2}^\perp$.

If $\mathbf{U} = \mathbf{0}$, then (4.33) holds due to Corollary 4.8.

If $\mathbf{U} \neq \mathbf{0}$, we can obtain from Corollary 4.10 that there exist $\mathbf{V}_M = (\mathbf{v}_{M_1}, q_{M_2}) \in H_{M_1} \times H_{M_2}$ and a positive constant c such that

$$\|\mathbf{U}_M\|_{\mathcal{H}^1} \leq c \frac{\operatorname{Re}\{A(\mathbf{U}_M, \mathbf{V}_M)\}}{\|\mathbf{V}_M\|_{\mathcal{H}^1}} \quad (4.34)$$

with $\|\mathbf{U}_M\|_{\mathcal{H}^1} = \|\mathbf{V}_M\|_{\mathcal{H}^1}$ after scaling. Now we define $\mathbf{V} = \mathbf{V}_M + \mathbf{U}_M^\perp$. Then (4.34) together with Corollary 4.8 give

$$\begin{aligned}\|\mathbf{U}_M\|_{\mathcal{H}^1}^2 + \|\mathbf{U}_M^\perp\|_{\mathcal{H}^1}^2 &\leq c(\operatorname{Re}\{A(\mathbf{U}_M, \mathbf{V}_M)\} + \operatorname{Re}\{A^N(\mathbf{U}_M^\perp, \mathbf{U}_M^\perp)\}) \\ &= c|A^N(\mathbf{U}, \mathbf{V})| + c|A^N(\mathbf{U}_M, \mathbf{U}_M^\perp)| + c|A^N(\mathbf{U}_M^\perp, \mathbf{V}_M)| \\ &\quad + c|b(p_{M_2}, q_{M_2}) - b^N(p_{M_2}, q_{M_2})|. \quad (4.35)\end{aligned}$$

According to Theorem 4.1, we obtain

$$\begin{aligned}|b(p_{M_2}, q_{M_2}) - b^N(p_{M_2}, q_{M_2})| &\leq \|(S - S^N)p_{M_2}\|_{H^{-3/2}(\Gamma_R)} \|q_{M_2}\|_{H^{3/2}(\Gamma_R)} \\ &\leq c \frac{\epsilon(N, p_M)}{N^2} \|p_{M_2}\|_{H^{3/2}(\Gamma_R)} \|q_{M_2}\|_{H^{3/2}(\Gamma_R)} \\ &\leq cN^{-2} \|p_{M_2}\|_{H^2(\Omega_R)} \|q_{M_2}\|_{H^2(\Omega_R)} \\ &\leq c_1 N^{-2} M_2 \gamma_0 \|\mathbf{U}_M\|_{\mathcal{H}^1}^2. \quad (4.36)\end{aligned}$$

Additionally, we have that

$$|A^N(\mathbf{U}_M, \mathbf{U}_M^\perp)| \leq |A(\mathbf{U}_M, \mathbf{U}_M^\perp)| + |b(p_{M_2}, p_{M_2}^\perp) - b^N(p_{M_2}, q_{M_2}^\perp)|. \quad (4.37)$$

Then Theorem 4.1 gives

$$\begin{aligned} |b(p_{M_2}, p_{M_2}^\perp) - b^N(p_{M_2}, q_{M_2}^\perp)| &\leq \|(S - S^N)p_{M_2}\|_{H^{-1/2}(\Gamma_R)} \|q_{M_2}^\perp\|_{H^{1/2}(\Gamma_R)} \\ &\leq c \frac{\epsilon(N, p_M)}{N} \|p_{M_2}\|_{H^{3/2}(\Gamma_R)} \|q_{M_2}^\perp\|_{H^{1/2}(\Gamma_R)} \\ &\leq cN^{-1} \|p_{M_2}\|_{H^2(\Omega_R)} \|q_{M_2}^\perp\|_{H^1(\Omega_R)} \\ &\leq c_2 N^{-1} M_2^{1/2} \gamma_0^{1/2} \|\mathbf{U}_M\|_{\mathcal{H}^1} \|\mathbf{U}_M^\perp\|_{\mathcal{H}^1}. \end{aligned} \quad (4.38)$$

Therefore, Corollary, (4.37), (4.38) and the arithmetic-geometric mean inequality lead to

$$|A^N(\mathbf{U}_M, \mathbf{U}_M^\perp)| \leq (c_3 \gamma_0^{1/4-\varepsilon/2} M_2^{\varepsilon/2-1/4} + c_4 N^{-1} M_2^{1/2} \gamma_0^{1/2}) (\|\mathbf{U}_M\|_{\mathcal{H}^1}^2 + \|\mathbf{U}_M^\perp\|_{\mathcal{H}^1}^2). \quad (4.39)$$

Similarly, we have

$$|A^N(\mathbf{U}_M^\perp, \mathbf{V}_M)| \leq (c_5 \gamma_0^{1/4-\varepsilon/2} M_2^{\varepsilon/2-1/4} + c_6 N^{-1} M_2^{1/2} \gamma_0^{1/2}) (\|\mathbf{U}_M\|_{\mathcal{H}^1}^2 + \|\mathbf{U}_M^\perp\|_{\mathcal{H}^1}^2). \quad (4.40)$$

Here, we need that $M_1, M_2 \geq M_0$ where $M_0 > 0$ satisfies (4.25), (4.26) and (4.32). In additionally, we suppose that $M_0 > 0$ and $N_0 \geq 0$ are large enough such that

$$1 - c_1 N_0^{-2} M_0 \gamma_0 - (c_3 + c_5) \gamma_0^{1/4-\varepsilon/2} M_0^{\varepsilon/2-1/4} - (c_5 + c_6) N_0^{-1} M_0^{1/2} \gamma_0^{1/2} > \frac{1}{2}, \quad (4.41)$$

which further implies that there is a positive constant c such that

$$\|\mathbf{U}_M\|_{\mathcal{H}^1}^2 + \|\mathbf{U}_M^\perp\|_{\mathcal{H}^1}^2 \leq c |A^N(\mathbf{U}, \mathbf{V})|. \quad (4.42)$$

Now, since

$$\begin{aligned} \alpha_0 \|\mathbf{u}\|_{(H^1\Omega)^2}^2 + \|p\|_{H^1(\Omega_R)}^2 &\leq \|\mathbf{u}\|_1^2 + \|p\|_2^2 \\ &= \|\mathbf{u}_{M_1}\|_1^2 + \|\mathbf{u}_{M_1}^\perp\|_1^2 + \|p_{M_2}\|_2^2 + \|p_{M_2}^\perp\|_2^2 \\ &\leq \beta_0 (\|\mathbf{u}_{M_1}\|_{(H^1\Omega)^2}^2 + \|\mathbf{u}_{M_1}^\perp\|_{(H^1\Omega)^2}^2) + \gamma_0 (\|p_{M_2}\|_{H^1(\Omega_R)}^2 + \|p_{M_2}^\perp\|_{H^1(\Omega_R)}^2) \\ &\leq \max\{\beta_0, \gamma_0\} (\|\mathbf{U}_M\|_{\mathcal{H}^1}^2 + \|\mathbf{U}_M^\perp\|_{\mathcal{H}^1}^2), \end{aligned}$$

we have

$$\|\mathbf{U}\|_{\mathcal{H}^1}^2 \leq \frac{\max\{\beta_0, \gamma_0\}}{\min\{1, \alpha_0\}} (\|\mathbf{U}_M\|_{\mathcal{H}^1}^2 + \|\mathbf{U}_M^\perp\|_{\mathcal{H}^1}^2). \quad (4.43)$$

Thus,

$$\|\mathbf{U}\|_{\mathcal{H}^1}^2 \leq c |A^N(\mathbf{U}, \mathbf{V})| \quad (4.44)$$

for some constant $c > 0$. Finally, we obtain from (4.43) similarly that

$$\begin{aligned} \|\mathbf{V}\|_{\mathcal{H}^1} &\leq \sqrt{\frac{\max\{\beta_0, \gamma_0\}}{\min\{1, \alpha_0\}}} (\|\mathbf{V}_M\|_{\mathcal{H}^1}^2 + \|\mathbf{U}_M^\perp\|_{\mathcal{H}^1}^2) \\ &= \sqrt{\frac{\max\{\beta_0, \gamma_0\}}{\min\{1, \alpha_0\}}} (\|\mathbf{U}_M\|_{\mathcal{H}^1}^2 + \|\mathbf{U}_M^\perp\|_{\mathcal{H}^1}^2) \\ &\leq \sqrt{\frac{\max\{\beta_0, \gamma_0\}}{\min\{1, \alpha_0\}}} \left(\frac{1}{\alpha_0} \|\mathbf{u}\|_1^2 + \|p\|_2^2 \right) \\ &\leq \sqrt{\frac{\max\{\beta_0, \gamma_0\}}{\min\{1, \alpha_0\}}} \max\left\{ \frac{\beta_0}{\alpha_0}, \gamma_0 \right\} \|\mathbf{U}\|_{\mathcal{H}^1} \end{aligned} \quad (4.45)$$

Therefore, (4.44) and (4.45) give

$$\begin{aligned}\|\mathbf{U}\|_{\mathcal{H}^1} &\leq c \frac{|A^N(\mathbf{U}, \mathbf{V})|}{\|\mathbf{V}\|_{\mathcal{H}^1}} \\ &\leq c \sup_{(\mathbf{0},0) \neq \mathbf{V} \in \mathcal{H}_h} \frac{|A^N(\mathbf{U}, \mathbf{V})|}{\|\mathbf{V}\|_{\mathcal{H}^1}},\end{aligned}$$

where $c > 0$ is a constant. This completes the proof. \square

According to Theorem 4.3, Theorem 4.11 and the Fredholm alternative theorem, we are let to the following result.

Theorem 4.12. *Let the surface Γ and the material parameter (μ, λ, ρ) be such that there are no traction free solutions, then there exists a constant $N_0 \geq 0$ such that the modified variational equation (4.6) admits a unique solution $(\mathbf{u}_N, p_N) \in \mathcal{H}^1$ for $N \geq N_0$.*

5 Finite element analysis

Our main goal in this section is to establishing a priori error estimates ([16, 19, 20, 27]) for the finite element solution of (4.6) in terms of the finite element mesh size h and the truncation order N in appropriate Sobolev spaces. It has been known that numerical errors induced from the truncation of DtN mapping are exponentially decaying ([36]), and thanks to the error estimate (4.5), we are able to derive a new upper bound of numerical errors indicating such effect explicitly.

5.1 Galerkin formulation

Let $\mathcal{H}_h = (\mathbf{S}_h, S'_h) \subset \mathcal{H}^1$ be the standard finite element space. Now we consider the Galerkin formulation of (4.6): Given p^{inc} , find $\mathbf{U}_h = (\mathbf{u}_h, p_h) \in \mathcal{H}_h$, $\mathbf{u}_h = (u_x^h, u_y^h)$ such that

$$A^N(\mathbf{U}_h, \mathbf{V}_h) = \ell(\mathbf{V}_h), \quad \forall \mathbf{V}_h = (\mathbf{v}_h, q_h) \in \mathcal{H}_h, \quad \mathbf{v}_h = (v_x^h, v_y^h). \quad (5.1)$$

It can be shown ([28]) that the discrete sesquilinear form $A^N(\cdot, \cdot)$ satisfies the BBL-condition as implication of the following:

Gårding's inequality + Uniqueness + Approximation property of $\mathcal{H}_h \Rightarrow$ BBL-condition.

Theorem 5.1. *Let the surface Γ and the material parameter (μ, λ, ρ) be such that there are no traction free solutions and suppose that the finite element space $\mathcal{H}_h \subset \mathcal{H}^1$ satisfies the standard approximation property, then there exist constants $N_0 \geq 0$ and $h_0 > 0$ such that $A^N(\cdot, \cdot)$ for $0 < h \leq h_0, N \geq N_0$ satisfies the BBL condition in the form*

$$\sup_{(\mathbf{0},0) \neq \mathbf{W}_h \in \mathcal{H}_h} \frac{|A^N(\mathbf{V}_h, \mathbf{W}_h)|}{\|\mathbf{W}_h\|_{\mathcal{H}^1}} \geq \gamma \|\mathbf{V}_h\|_{\mathcal{H}^1}, \quad \forall \mathbf{V}_h \in \mathcal{H}_h. \quad (5.2)$$

Here $\gamma > 0$ is the inf-sup constant independent of h .

From the BBL condition (5.2), we are allowed to derive a priori error estimates for the finite element solution $\mathbf{U}_h \in \mathcal{H}_h$.

5.2 Asymptotic error estimates

In this subsection, we mainly derive a reasonable a priori error estimates on appropriate Sobolev spaces including error effects of both the numerical discretization and the truncation of infinite series. We first establish an upper bound of numerical errors analogous to the well-known Céa's lemma in the positive definite case.

Theorem 5.2. *There exist constants $h_0 > 0$ and $N_0 \geq 0$ such that for any $h \in (0, h_0]$ and $N \geq N_0$*

$$\|\mathbf{U} - \mathbf{U}_h\|_{\mathcal{H}^1} \leq c \left\{ \inf_{\mathbf{V}_h \in \mathcal{H}_h} \|\mathbf{U} - \mathbf{V}_h\|_{\mathcal{H}^1} + \sup_{0 \neq w_2 \in S'_h} \frac{|(b(p, w_2) - b^N(p, w_2))|}{\|w_2\|_{H^1(\Omega_R)}} \right\} \quad (5.3)$$

where $c > 0$ is a constant independent of h and N .

Proof. From the BBL condition (5.2), we know that

$$\gamma \|\mathbf{U}_h - \mathbf{V}_h\|_{\mathcal{H}^1} \leq \sup_{(\mathbf{0}, 0) \neq \mathbf{W}_h \in \mathcal{H}_h} \frac{|A^N(\mathbf{U}_h - \mathbf{V}_h, \mathbf{W}_h)|}{\|\mathbf{W}_h\|_{\mathcal{H}^1}}, \quad \forall \mathbf{V}_h \in \mathcal{H}_h. \quad (5.4)$$

by replacing \mathbf{V}_h with $\mathbf{U}_h - \mathbf{V}_h \in \mathcal{H}_h$. According to (3.13) and (5.1), we have

$$B(\mathbf{U}, \mathbf{W}_h) + b^N(p, w_2) = \ell(\mathbf{W}_h) + b^N(p, w_2) - b(p, w_2), \quad \forall \mathbf{W}_h = (\mathbf{w}_1, w_2) \in \mathcal{H}_h, \quad (5.5)$$

and

$$B(\mathbf{U}_h, \mathbf{W}_h) + b^N(p_h, w_2) = \ell(\mathbf{W}_h), \quad \forall \mathbf{W}_h = (\mathbf{w}_1, w_2) \in \mathcal{H}_h, \quad (5.6)$$

respectively. Therefore, subtracting (5.5) from (5.6) leads to

$$A^N(\mathbf{U}_h - \mathbf{U}, \mathbf{W}_h) = b(p, w_2) - b^N(p, w_2). \quad (5.7)$$

Thus, by (5.7), the inequality (5.4) implies that

$$\begin{aligned} \gamma \|\mathbf{U}_h - \mathbf{V}_h\|_{\mathcal{H}^1} &\leq \sup_{(\mathbf{0}, 0) \neq \mathbf{W}_h \in \mathcal{H}_h} \frac{|A^N(\mathbf{U}_h - \mathbf{V}_h, \mathbf{W}_h)|}{\|\mathbf{W}_h\|_{\mathcal{H}^1}} \\ &= \sup_{(\mathbf{0}, 0) \neq \mathbf{W}_h \in \mathcal{H}_h} \frac{|A^N(\mathbf{U} - \mathbf{V}_h, \mathbf{W}_h) + A^N(\mathbf{U}_h - \mathbf{U}, \mathbf{W}_h)|}{\|\mathbf{W}_h\|_{\mathcal{H}^1}} \\ &= \sup_{(\mathbf{0}, 0) \neq \mathbf{W}_h \in \mathcal{H}_h} \frac{|A^N(\mathbf{U} - \mathbf{V}_h, \mathbf{W}_h) + b(p, w_2) - b^N(p, w_2)|}{\|\mathbf{W}_h\|_{\mathcal{H}^1}} \\ &\leq \sup_{(\mathbf{0}, 0) \neq \mathbf{W}_h \in \mathcal{H}_h} \frac{|A^N(\mathbf{U} - \mathbf{V}_h, \mathbf{W}_h)|}{\|\mathbf{W}_h\|_{\mathcal{H}^1}} + \sup_{(\mathbf{0}, 0) \neq \mathbf{W}_h \in \mathcal{H}_h} \frac{|b(p, w_2) - b^N(p, w_2)|}{\|\mathbf{W}_h\|_{\mathcal{H}^1}} \\ &\leq c \|\mathbf{U} - \mathbf{V}_h\|_{\mathcal{H}^1} + \sup_{0 \neq w_2 \in S'_h} \frac{|b(p, w_2) - b^N(p, w_2)|}{\|w_2\|_{H^1(\Omega_R)}}, \end{aligned} \quad (5.8)$$

where c is a positive constant. Consequently, the triangular inequality and the formulation (5.8) yield, $\forall \mathbf{V}_h \in \mathcal{H}_h$,

$$\begin{aligned} \|\mathbf{U} - \mathbf{U}_h\|_{\mathcal{H}^1} &\leq \|\mathbf{U} - \mathbf{V}_h\|_{\mathcal{H}^1} + \|\mathbf{U}_h - \mathbf{V}_h\|_{\mathcal{H}^1} \\ &\leq (1 + \frac{c}{\gamma}) \|\mathbf{U} - \mathbf{V}_h\|_{\mathcal{H}^1} + \frac{1}{\gamma} \sup_{0 \neq w_2 \in S'_h} \frac{|b(p, w_2) - b^N(p, w_2)|}{\|w_2\|_{H^1(\Omega_R)}} \end{aligned}$$

and this further leads to

$$\|\mathbf{U} - \mathbf{U}_h\|_{\mathcal{H}^1} \leq c \left\{ \inf_{\mathbf{V}_h \in \mathcal{H}_h} \|\mathbf{U} - \mathbf{V}_h\|_{\mathcal{H}^1} + \sup_{0 \neq w_2 \in S'_h} \frac{|b(p, w_2) - b^N(p, w_2)|}{\|w_2\|_{H^1(\Omega_R)}} \right\}$$

where $c > 0$ is a constant independent of h and N . This completes the proof. \square

From the estimate (5.3), we can see that the numerical errors is constructed by two single terms. We study the first term correlated with the finite element meshsize h by using the approximation theory, and the second term dependent on the truncation order N by employing the point estimate (4.5). Let $\mathcal{H}^t = (H^t(\Omega))^2 \times H^t(\Omega_R)$. In the following, starting with the estimate (5.3), we derive a priori error estimates measured in \mathcal{H}^1 -norm and \mathcal{H}^0 -norm respectively to conclude this section.

Theorem 5.3. *Suppose that $\mathbf{U} \in \mathcal{H}^t$ for $2 \leq t \in \mathbb{R}$. Then there exist constants $h_0 > 0$ and $N_0 \geq 0$ such that for any $h \in (0, h_0]$ and $N \geq N_0$*

$$\|\mathbf{U} - \mathbf{U}_h\|_{\mathcal{H}^1} \leq c \{h^{t-1} \|\mathbf{U}\|_{\mathcal{H}^t} + q^N \|p\|_{H^t(\Omega_R)}\}, \quad (5.9)$$

where $c > 0$ and $0 < q < 1$ are constants independent of h and N .

Proof. We have known from Theorem 5.2 that

$$\|\mathbf{U} - \mathbf{U}_h\|_{\mathcal{H}^1} \leq c \left\{ \inf_{\mathbf{V}_h \in \mathcal{H}_h} \|\mathbf{U} - \mathbf{V}_h\|_{\mathcal{H}^1} + \sup_{0 \neq w_2 \in S'_h} \frac{|b(p, w_2) - b^N(p, w_2)|}{\|w_2\|_{H^1(\Omega_R)}} \right\}, \quad (5.10)$$

where $c > 0$ is a positive constant. The approximation property of the finite element space \mathcal{H}_h gives

$$\inf_{\mathbf{V}_h \in \mathcal{H}_h} \|\mathbf{U} - \mathbf{V}_h\|_{\mathcal{H}^1} \leq ch^{t-1} \|\mathbf{U}\|_{\mathcal{H}^t}, \quad (5.11)$$

where $c > 0$ is a positive constant independent of h . We now consider the second term in (5.10). The trace theorem implies that there exists a bounded linear operator $\gamma : H^1(\Omega_R) \rightarrow H^{1/2}(\Gamma_R)$ such that

$$\frac{|b(p, w_2) - b^N(p, w_2)|}{\|w_2\|_{H^1(\Omega_R)}} = \frac{|\langle (S - S^N)\gamma p, \gamma w_2 \rangle_{\Gamma_R}|}{\|w_2\|_{H^1(\Omega_R)}} = \frac{|\langle \gamma^*(S - S^N)\gamma p, w_2 \rangle_{\Omega_R}|}{\|w_2\|_{H^1(\Omega_R)}}, \quad (5.12)$$

where $\langle \cdot, \cdot \rangle_{\Gamma_R}$ is the standard L^2 duality pairing between $H^{-1/2}(\Gamma_R)$ and $H^{1/2}(\Gamma_R)$, and $\gamma^* : H^{-1/2}(\Gamma_R) \rightarrow (H^1(\Omega_R))'$ is the adjoint operator of γ . Therefore, we have

$$\begin{aligned} \sup_{0 \neq w_2 \in S'_h} \frac{|b(p, w_2) - b^N(p, w_2)|}{\|w_2\|_{H^1(\Omega_R)}} &= \sup_{0 \neq w_2 \in S'_h} \frac{|\langle \gamma^*(S - S^N)\gamma p, w_2 \rangle_{\Omega_R}|}{\|w_2\|_{H^1(\Omega_R)}} \\ &= \|\gamma^*(S - S^N)\gamma p\|_{(H^1(\Omega_R))'} \\ &\leq c\|(S - S^N)\gamma p\|_{H^{-1/2}(\Gamma_R)} \\ &\leq cq^N \|\gamma p\|_{H^{t-1/2}(\Gamma_R)} \\ &\leq cq^N \|p\|_{H^t(\Omega_R)} \end{aligned} \quad (5.13)$$

because of Theorem 4.2 and the boundedness of operators γ and γ^* . The proof is hence established by following a combination of (5.11) and (5.13). \square

We now extend the error estimate in the energy space to the one measured in the $\mathcal{H}^0 = (L_2(\Omega))^2 \times L_2(\Omega_R)$ space.

Theorem 5.4. *Suppose that $\mathbf{U} \in \mathcal{H}^t$ for $2 \leq t \in \mathbb{R}$. Then there exist constants $h_0 > 0$ and $N_0 \geq 0$ such that for any $h \in (0, h_0]$ and $N \geq N_0$*

$$\|\mathbf{U} - \mathbf{U}_h\|_{\mathcal{H}^0} \leq c \{h^t \|\mathbf{U}\|_{\mathcal{H}^t} + q^N \|p\|_{H^t(\Omega_R)}\} \quad (5.14)$$

where $c > 0$ and $0 < q < 1$ are constants independent of h and N .

Proof. Let $\mathbf{E} = (\mathbf{e}_1, e_2) = \mathbf{U} - \mathbf{U}_h$ be the finite element error. Then by (5.7) in Theorem 5.2, we have

$$B(\mathbf{E}, \mathbf{V}_h) + b^N(e_2, v_2) + b(p, v_2) - b^N(p, v_2) = 0, \quad \forall \mathbf{V}_h = (\mathbf{v}_1, v_2) \in \mathcal{H}_h. \quad (5.15)$$

Now, we consider the following boundary value problem: Find $\mathbf{W} = (\mathbf{w}_1, w_2) \in (C^2(\Omega) \cap C^1(\bar{\Omega}))^2 \times (C^2(\Omega_R) \cap C^1(\bar{\Omega}_R))$ satisfying

$$\Delta^* \mathbf{w}_1 + \rho \omega^2 \mathbf{w}_1 = \mathbf{e}_1 \quad \text{in } \Omega, \quad (5.16)$$

$$\Delta w_2 + k^2 w_2 = e_2 \quad \text{in } \Omega_R, \quad (5.17)$$

$$\rho_f \omega^2 \mathbf{w}_1 \cdot \mathbf{n} = \frac{\partial w_2}{\partial n} \quad \text{on } \Gamma, \quad (5.18)$$

$$\mathbf{T} \mathbf{w}_1 = -w_2 \mathbf{n} \quad \text{on } \Gamma, \quad (5.19)$$

$$\frac{\partial w_2}{\partial n} = S w_2 \quad \text{on } \Gamma_R. \quad (5.20)$$

Let $\mathbf{W} = (\mathbf{w}_1, w_2)$ be a weak solution of nonlocal boundary value problem (5.16)-(5.20), hence \mathbf{W} satisfies

$$A^N(\mathbf{V}, \mathbf{W}) + b(v_2, w_2) - b^N(v_2, w_2) = (\mathbf{w}_1, \mathbf{e}_1)_{(L^2(\Omega))^2} + (w_2, e_2)_{L^2(\Omega_R)}, \quad \forall \mathbf{V} = (\mathbf{v}_1, v_2) \in \mathcal{H}^1. \quad (5.21)$$

Replacing \mathbf{V} by \mathbf{E} in (5.21) gives

$$A^N(\mathbf{E}, \mathbf{W}) + b(e_2, w_2) - b^N(e_2, w_2) = (\mathbf{e}_1, \mathbf{e}_1)_{(L^2(\Omega))^2} + (e_2, e_2)_{L^2(\Omega_R)} = \|\mathbf{E}\|_{\mathcal{H}^0}^2. \quad (5.22)$$

Then subtracting (5.22) from (5.15) leads to, for $\forall \mathbf{V}_h \in \mathcal{H}_h$,

$$\|\mathbf{E}\|_{\mathcal{H}^0}^2 = A^N(\mathbf{E}, \mathbf{W} - \mathbf{V}_h) + b(e_2, w_2) - b^N(e_2, w_2) + b^N(p, v_2) - b(p, v_2). \quad (5.23)$$

Theorem 4.4, the approximation property of \mathcal{H}_h and the regularity theory imply that

$$\begin{aligned} |A^N(\mathbf{E}, \mathbf{W} - \mathbf{V}_h)| &\leq c \|\mathbf{E}\|_{\mathcal{H}^1} \|\mathbf{W} - \mathbf{V}_h\|_{\mathcal{H}^1} \\ &\leq ch \|\mathbf{E}\|_{\mathcal{H}^1} \|\mathbf{W}\|_{\mathcal{H}^2} \\ &\leq ch \|\mathbf{E}\|_{\mathcal{H}^1} \|\mathbf{E}\|_{\mathcal{H}^0}, \end{aligned} \quad (5.24)$$

where $c > 0$ is a constant. Following the same argument in Theorem 5.3 and choosing $t = 2$, we arrive at, by the regularity theory,

$$\begin{aligned} |b(e_2, w_2) - b^N(e_2, w_2)| &\leq \|\gamma^*(S - S^N)\gamma e_2\|_{(H^2(\Omega_R))'} \|w_2\|_{H^2(\Omega_R)} \\ &\leq c \|(S - S^N)\gamma e_2\|_{H^{-3/2}(\Gamma_R)} \|\mathbf{W}\|_{\mathcal{H}^2} \\ &\leq cq^N \|\gamma e_2\|_{H^{1/2}(\Gamma_R)} \|\mathbf{E}\|_{\mathcal{H}^0} \\ &\leq cq^N \|e_2\|_{H^1(\Omega_R)} \|\mathbf{E}\|_{\mathcal{H}^0} \end{aligned} \quad (5.25)$$

Similarly, we also have

$$\begin{aligned} |b^N(p, w_2 - v_2) - b(p, w_2 - v_2)| &\leq \|\gamma^*(S - S^N)\gamma p\|_{(H^1(\Omega_R))'} \|w_2 - v_2\|_{H^1(\Omega_R)} \\ &\leq c \|(S - S^N)\gamma p\|_{H^{-1/2}(\Gamma_R)} \|\mathbf{W} - \mathbf{V}_h\|_{\mathcal{H}^1} \\ &\leq chq^N \|\gamma p\|_{H^{t-1/2}(\Gamma_R)} \|\mathbf{W}\|_{\mathcal{H}^2} \\ &\leq chq^N \|p\|_{H^t(\Omega_R)} \|\mathbf{E}\|_{\mathcal{H}^0}. \end{aligned}$$

and

$$\begin{aligned} |b(p, w_2) - b^N(p, w_2)| &\leq \|\gamma^*(S - S^N)\gamma p\|_{(H^2(\Omega_R))'} \|w_2\|_{H^2(\Omega_R)} \\ &\leq c \|(S - S^N)\gamma p\|_{H^{-3/2}(\Gamma_R)} \|\mathbf{W}\|_{\mathcal{H}^2} \\ &\leq cq^N \|\gamma p\|_{H^{t-1/2}(\Gamma_R)} \|\mathbf{E}\|_{\mathcal{H}^0} \\ &\leq cq^N \|p\|_{H^t(\Omega_R)} \|\mathbf{E}\|_{\mathcal{H}^0}, \end{aligned}$$

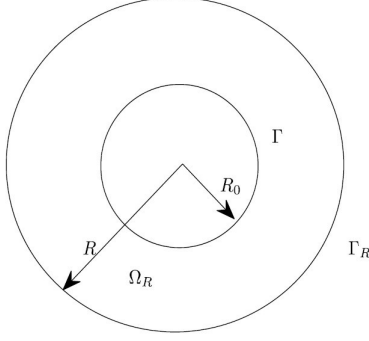


Figure 2: The computational domain of the model problem.

Thus, by the triangular inequality we know

$$\begin{aligned} |b^N(p, v_2) - b(p, v_2)| &\leq |b^N(p, w_2 - v_2) - b(p, w_2 - v_2)| + |b^N(p, w_2) - b(p, w_2)| \\ &\leq c_1 h q^N \|p\|_{H^t(\Omega_R)} \|\mathbf{E}\|_{\mathcal{H}^0} + c_2 q^N \|p\|_{H^t(\Omega_R)} \|\mathbf{E}\|_{\mathcal{H}^0}. \end{aligned} \quad (5.26)$$

Therefore, by the combination of the inequalities (5.24)-(5.26) and (5.9) we have

$$\begin{aligned} \|\mathbf{E}\|_{\mathcal{H}^0} &= A^N(\mathbf{E}, \mathbf{W} - \mathbf{V}_h) + b(e_2, w_2) - b^N(e_2, w_2) + b^N(p, v_2) - b(p, v_2) \\ &\leq ch \{h^{t-1} \|\mathbf{U}\|_{\mathcal{H}^t} + q^N \|p\|_{H^t(\Omega_R)}\} + cq^N \{h^{t-1} \|\mathbf{U}\|_{\mathcal{H}^t} + q^N \|p\|_{H^t(\Omega_R)}\} \\ &\quad + chq^N \|p\|_{H^t(\Omega_R)} + cq^N \|p\|_{H^t(\Omega_R)}. \end{aligned}$$

Finally, according to the fact that $h \in (0, h_0]$ and $N \geq N_0$, we arrive at (5.14). \square

6 Numerical experiments

In this section, we present several numerical tests to validate our theoretical results, and, unless otherwise stated, we always take the parameters $\omega = 1$, $\mu = 1$, $\lambda = 1$, $\rho = 1$, $\rho_f = 1$ and $R_0 = 1$. We first introduce a model problem whose analytical solution is available so that we are able to evaluate the accuracy of the numerical solution. We consider the scattering of a plane incident wave $p^{inc} = e^{ikx \cdot d}$ with direction $d = (1, 0)$ by a disc-shaped elastic body of radius R_0 (see Figure 2). For the exact solution of this model problem, please see the Appendix. In numerical computations, the computational domain Ω and Ω_R are discretized by uniform triangle elements and we employ piecewise linear basis functions $\{\varphi_i\}_{i=1}^{i=N_1}$ and $\{\psi_i\}_{i=1}^{i=N_2}$ in Ω and Ω_R , respectively, to construct the finite element space \mathcal{H}_h . Here, N_1 and N_2 are the total number of elements in Ω and Ω_R , respectively. To find the finite element solution of (5.1), one needs to compute the integrals

$$\int_{\Gamma_R} (S^N \psi_j) \overline{\psi_i} ds, \quad (6.1)$$

for those ψ_i and ψ_j not vanishing on Γ_R . For the sake of simplicity, (6.1) can be approximated by

$$\int_{\Gamma_R} (S^N \psi_j) \overline{\psi_i} ds \approx \int_{\Gamma_R} (S^N \zeta_j) \overline{\zeta_i} ds, \quad (6.2)$$

where $\{\zeta_i\}$ are the corresponding piecewise linear basis functions in terms of θ on Γ_R . Thus, the computation of integrals (6.1) amounts to evaluating a series as

$$\int_{\Gamma_R} (S^N \psi_j) \overline{\psi_i} ds \approx \sum_{n=0}^N \frac{k R H_n^{(1)'}(kR)}{\pi H_n^{(1)}(kR)} \int_0^{2\pi} \int_0^{2\pi} \zeta_j(R, \phi) \overline{\zeta_i(R, \theta)} \cos(n(\theta - \phi)) d\theta d\phi,$$

or

$$\int_{\Gamma_R} (S^N \psi_j) \overline{\psi_i} ds \approx \sum_{n=-N}^N \frac{k R H_n^{(1)'}(kR)}{\pi H_n^{(1)}(kR)} \int_0^{2\pi} \zeta_j(R, \phi) e^{-in\phi} d\phi \int_0^{2\pi} \overline{\zeta_i(R, \theta)} e^{in\theta} d\theta.$$

In the numerical experiments, we firstly check the accuracy of the DtN-FEM. We choose $R = 2$, $N = 20$ and consider three different wave numbers $k = 1, 2$ and 4 . Figure 3 shows the exact and numerical solutions of the elastic displacement $\mathbf{u} = (u_x, u_y)$ in the solid and the acoustic scattered field p in the fluid for the problem (2.1)-(2.5) with $k = 1$ and meshsize $h = 0.1076$. We can observe that the numerical solutions are in a perfect agreement with the exact series solutions from the qualitative point of view. According to Theorems 5.3 and 5.4, as the truncation order N of the DtN mapping is large enough that the domain discretization error is dominant, we should be able to observe that

$$\|\mathbf{U} - \mathbf{U}_h\|_{\mathcal{H}^0} = O(h^2), \quad \|\mathbf{U} - \mathbf{U}_h\|_{\mathcal{H}^1} = O(h)$$

for the finite element space \mathcal{H}_h . In Figure 4, we show the log-log plot of errors $\mathbf{U} - \mathbf{U}_h$ measured in \mathcal{H}^0 and \mathcal{H}^1 -norms with respect to $1/h$ which verifies the convergence order of $O(h^2)$ and $O(h)$. Finally, we are concerned with the effects of truncation order N on the total numerical errors. We choose $k = 1$ and compute the numerical errors measured in \mathcal{H}^0 -norm for three different meshsizes $h = 0.4304, 0.2151$ and 0.1076 , respectively. The log-log plots of errors are presented in Figure 5 showing that the errors due to the truncation of the DtN mapping decay extremely fast, arriving at the low bound correlating to each finite element mesh size.

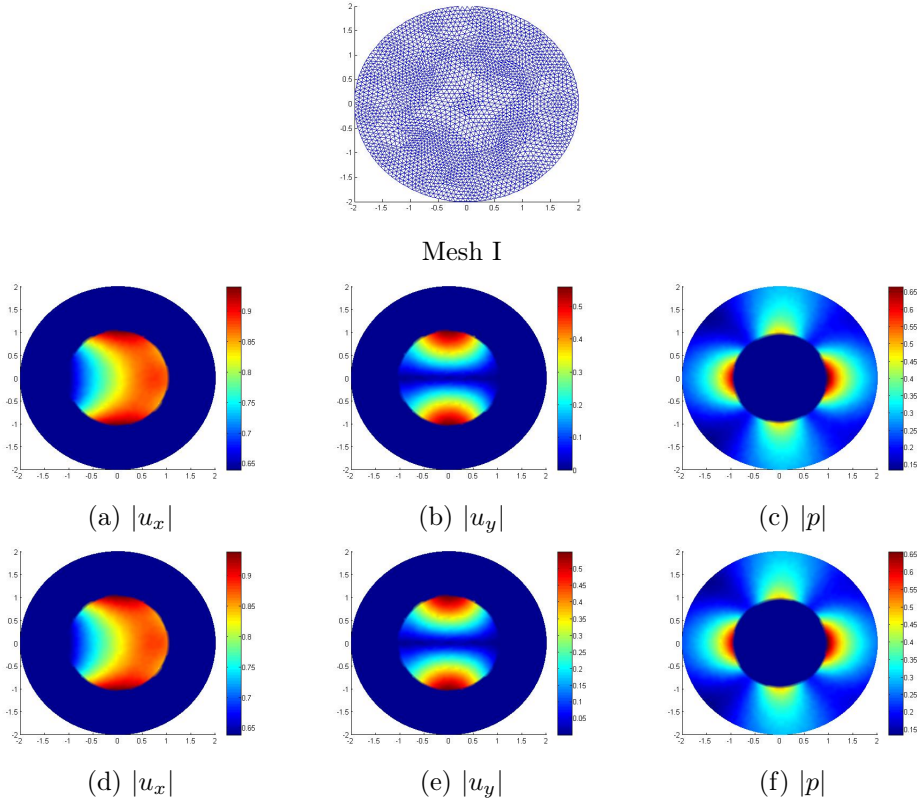


Figure 3: Exact solutions (a,b,c) and numerical solutions of DtN-FEM (d,e,f) using Mesh I.

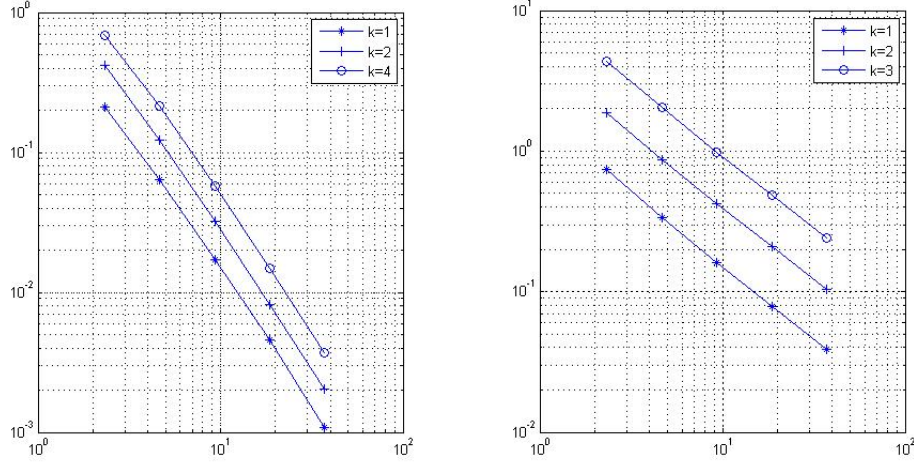


Figure 4: Log-log plots for numerical errors (vertical) of \mathbf{U} vs. $1/h$ (horizontal). Left: \mathcal{H}^0 -norm; right: \mathcal{H}^1 -norm.

7 Conclusion

In this paper, we consider the DtN-FEM solving the two dimensional FSI problem in which the DtN mapping is defined by a Fourier series on a circle. Uniqueness and existence of solutions for the corresponding variational problem are established. We also consider a truncated variational problem due to the truncation of DtN mapping and its solvability results are given by using a novel proof technique. A priori error estimates involving the effects of both finite element meshsize and the truncation order of series for the proposed DtN-FEM are derived. The investigation strategy in this article will be extended to other scattering problems in the future.

Appendix

Here, we derive the exact solution for the considered model problem in Section 6. We can represent the solution of (2.1)-(2.5) as

$$\begin{aligned} p(r, \theta) &= \sum_{n=0}^{\infty} A_n H_n^{(1)}(kr) \cos(n\theta), \\ \mathbf{u} &= \nabla \varphi - \nabla \times \psi, \end{aligned}$$

with

$$\begin{aligned} \varphi(r, \theta) &= \sum_{n=0}^{\infty} B_n J_n(k_p r) \cos(n\theta), \\ \psi(r, \theta) &= \sum_{n=0}^{\infty} C_n J_n(k_s r) \sin(n\theta), \end{aligned}$$

where the coefficients A_n , B_n and C_n are to be determined. According to the transmission condition (2.3)-(2.4), we are able to obtain a linear system of equations as

$$[E_n^{ij}] X_n = [e_n^j], i, j = 1, 2, 3,$$

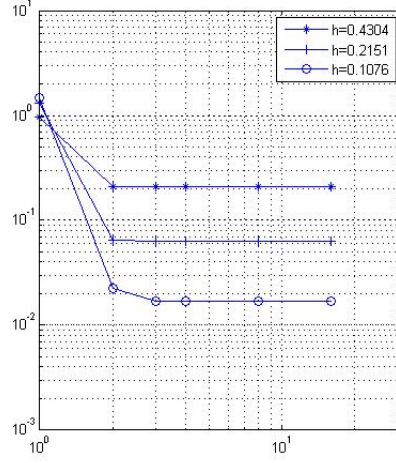


Figure 5: Log-log plot for numerical errors $\|\mathbf{U} - \mathbf{U}_h\|_{\mathcal{H}^0}$ (vertical) vs. N (horizontal).

where $X_n = (A_n, C_n, D_n)^T$. For the system matrix E_n and vector e_n , the elements (identified by the super-script) are computed as follows:

$$\begin{aligned}
E_n^{11} &= -H_{n-1}^{(1)}(kR_0) + \frac{n}{kR_0}H_n^{(1)}(kR_0), \\
E_n^{12} &= \frac{\rho_f\omega^2k_p}{k} \left[J_{n-1}(k_pR_0) - \frac{n}{k_pR_0}J_n(k_pR_0) \right], \\
E_n^{13} &= \frac{\rho_f\omega^2n}{kR_0}J_n(k_sR_0), \\
E_n^{21} &= 0, \\
E_n^{22} &= \frac{2\mu nk_p}{R_0}J_{n-1}(k_pR_0) - \frac{2\mu(n^2+n)}{R_0^2}J_n(k_pR_0), \\
E_n^{23} &= \frac{2\mu(n^2+n) - \mu k_s^2 R_0^2}{R_0^2}J_n(k_sR_0) - \frac{2\mu k_s}{R_0}J_{n-1}(k_sR_0), \\
E_n^{31} &= H_n^{(1)}(kR_0), \\
E_n^{32} &= \frac{2\mu(n^2+n) - \mu k_s^2 R_0^2}{R_0^2}J_n(k_pR_0) - \frac{2\mu k_p}{R_0}J_{n-1}(k_pR_0), \\
E_n^{33} &= \frac{2\mu nk_s}{R_0}J_{n-1}(k_sR_0) - \frac{2\mu(n^2+n)}{R_0^2}J_n(k_sR_0), \\
e_n^1 &= \epsilon_n i^n \left[J_{n-1}(kR_0) - \frac{n}{kR_0}J_n(kR_0) \right], \\
e_n^2 &= 0, \\
e_n^3 &= -\epsilon_n i^n J_n(kR_0).
\end{aligned}$$

Acknowledgments

The work of T. Yin is partially supported by the NSFC Grant (11371385). The work of L. Xu is partially supported by the NSFC Grant (11371385), the Start-up fund of Youth 1000 plan of China and

that of Youth 100 plan of Chongqing University.

References

- [1] M. Abramowitz, I. A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1965.
- [2] J.P. Berenger, A perfectly matched layer for the absorption of electromagnetic waves, J. Comput. Phys., 114 (2) (1994) 185-200.
- [3] J. Bielak, R.C. MacCamy, Symmetric finite element and boundary integral coupling methods for fluid-solid interaction, Q. Appl. Math. 49 (1991) 107-119.
- [4] J.H. Bramble, J.E. Pasciak, Analysis of a Cartesian PML approximation to acoustic scattering problems in \mathbb{R}^2 and \mathbb{R}^3 , J. Comput. Appl. Math. 247 (2013), 209-230.
- [5] Z. Chen, X. Liu, An Adaptive Perfectly Matched Layer Technique for Time-Harmonic Scattering Problems, SIAM J. Numer. Anal. 43 (2) (2006) 645-671.
- [6] C. Domínguez, E.P. Stephan, M. Maischak, A FE-BE coupling for a fluid-structure interaction problem: hierarchical a posteriori error estimates, Numer. Methods Partial Differential Equations 28 (2012) 1417-1439.
- [7] C. Domínguez, E.P. Stephan, M. Maischak, FE/BE coupling for an acoustic fluid-structure interaction problem. Residual a posteriori error estimates, Internat. J. Numer. Methods Engrg. 89 (2012) 299-322.
- [8] Q. Du, M.D. Gunzburger, L.S. Hou, J. Lee, Semidiscrete finite element approximations of a linear fluid-structure interaction problem, SIAM J. Numer. Anal. 42 (1) 1-29.
- [9] K. Feng, Finite element method and natural boundary reduction, in: Proceedings of the International Congress of Mathematicians, Warsaw, 1983, 1439-1453.
- [10] K. Feng, Asymptotic radiation conditions for reduced wave equation, J. Comput. Math. 2 (1984) 130-138.
- [11] L.C. Evans, Partial differential equations, volume 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1998.
- [12] G.N. Gatica, G.C. Hsiao, S. Meddahi, A residual-based a posteriori error estimator for a two-dimensional fluid-solid interaction problem, Numer. Math. 114 (2009) 63-106.
- [13] G.N. Gatica, G.C. Hsiao, S. Meddahi, A coupled mixed finite element method for the interaction problem between an electromagnetic field and an elastic body, SIAM J. Numer. Anal. 48 (4) (2010) 1338-1368.
- [14] G.N. Gatica, A. Márquez, S. Meddahi, Analysis of the coupling of primal and dual-mixed finite element methods for a two-dimensional fluid-solid interaction problem, SIAM J. Numer. Anal. 45 (5) (2007) 2072-2097.
- [15] G.N. Gatica, A. Márquez, S. Meddahi, Analysis of the coupling of Lagrange and Arnold-Falk-Winther finite elements for a fluid-solid interaction problem in three dimensions, SIAM J. Numer. Anal. 50 (3) (2012) 1648-1674.
- [16] H. Geng, T. Yin, L. Xu, F. Zeng, A priori error estimates of the DtN-FEM: acoustic transmission problems, submitted.

- [17] D. Givoli, Recent advances in the DtN FE method, *Arch. Comput. Methods Engrg.* 6 (2) (1999) 71-116.
- [18] M.J. Grote, J. Keller, On nonreflecting boundary conditions, *J. Comput. Phys.* 122 (1995) 231-243.
- [19] H. Han, X. Wu, Approximation of infinite boundary condition and its application to finite element methods, *J. Comput. Math.* 3 (1985) 179-192.
- [20] H. Han, X. Wu, The approximation of the exact boundary conditions at an artificial boundary for linear elastic equations and its application, *Math. Comp.* 59 (1992) 21-37.
- [21] H. Han, X. Wu, *Artificial boundary method*, Springer-Verlag Berlin Heidelberg, 2013.
- [22] I. Harari, T.J.R. Hughes, Analysis of continuous formulations underlying the computation of time-harmonic acoustics in exterior domains. *Comput. Methods Appl. Mech. Eng.* 97(1) (1992) 103-124.
- [23] C.O. Horgan, Korn's inequalities and their applications in continuum Mechanics, *SIAM Review*, 37 (4) (1995) 491-511.
- [24] G.C. Hsiao, On the boundary-fitted equation methods for fluid-structure interactions, *Problems and Methods in Mathematical Physics*, Teubner-Text zur Mathematik, 34 (1994) 79-88.
- [25] G.C. Hsiao, R.E. Kleinman, G.F. Roach, Weak solutions of fluid-solid interaction problems, *Math. Nachr.* 218 (2000) 139-163.
- [26] G.C. Hsiao, R.E. Kleinman, L.S. Schuetz, On variational formulations of boundary value problems for fluid-solid interactions, *Elastic Wave Propagation (Galway)*, North-Holland Ser. Appl. Math. Mech. , North-Holland, Amsterdam, 35 (1988) 312-326.
- [27] G.C. Hsiao, N. Nigam, J.E. Pasciak, L. Xu, Error analysis of the DtN-FEM for the scattering problem in acoustics via Fourier analysis, *J. Com. Appl. Math.* 235 (2011) 4949-4965.
- [28] G. Hsiao, W. Wendland, Boundary element methods: foundation and error analysis, in: E. Stein, R. de Borst, T.J.R. Hughes (Eds.), in: *Encyclopedia of Computational Mechanics*, vol. 1, John Wiley and Sons, Ltd., 2004, pp. 339-373.
- [29] T. Huttunen, J.P. Kaipio, P. Monk, An ultra-weak method for acoustic fluid-solid interaction, *J. Comput. Appl. Math.* 213 (2008) 166-185.
- [30] D.S. Jones, Low-frequency scattering by a body in lubricated contact, *Quart. J. Mech. Appl. Math.* 36 (1983) 111-137.
- [31] J. Keller, D. Givoli, Exact non-reflecting boundary conditions, *J. Comput. Phys.* 82 (1989) 172-192.
- [32] C.J. Luke, P.A. Martin, Fluid-solid interaction: acoustic scattering by a smooth elastic obstacle, *SIAM J. Appl. Math.* 55 (4) (1995) 904-922.
- [33] A. Márquez, S. Meddahi, V. Selgas, A new BEM-FEM coupling strategy for two-dimensional fluid-solid interaction problems, *J. Comput. Phys.* 199 (2004) 205-220.
- [34] J.M. Melenk, S. Sauter, Convergence analysis for finite element discretizations of the Helmholtz equation with Dirichlet-to-Neumann boundary conditions, *Math. Comp.* 79 (2010) 1871-1914.
- [35] D. Nicholls, N. Nigam, Exact non-reflecting boundary conditions on general domains, *J. Comput. Phys.* 194 (2004) 278-303.
- [36] D. Nicholls, N. Nigam, Error analysis of an enhanced DtN-FE method for exterior scattering problems, *Numer. Math.* 105 (2006) 267-298.

- [37] A. Schatz, An observation concerning Ritz-Galerkin methods with indefinite bilinear forms, *Math. Comp.* 28 (1974) 959-962.
- [38] T. Yin, G.C. Hsiao, L. Xu, Boundary integral equation methods for the two dimensional fluid-solid interaction problem, submitted.
- [39] T. Yin, G. Hu, L. Xu, B. Zhang, Near-field imaging of obstacles with the factorization method: fluid-solid interaction, *Inver. Prob.* 32 (2016) 015003.
- [40] T. Yin, A. Rathsfeld, L. Xu, A BIE-based DtN-FEM for fluid-solid interaction problems, to appear, *J. Comp. Math.*.